TOTALLY GEODESIC SUBALGEBRAS OF NILPOTENT LIE ALGEBRAS II

GRANT CAIRNS, ANA HINIĆ-GALIĆ, AND YURY NIKOLAYEVSKY

Abstract. We determine the maximal dimension of totally geodesic subalgebras of $\mathbb{N}$-graded filiform Lie algebras, and we show that these bounds are attained.

1. Introduction

Consider a finite-dimensional Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is equipped with an inner product $\langle\cdot,\cdot\rangle$ and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, we denote its orthogonal complement $\mathfrak{h}^\perp$. We say that $\mathfrak{h}$ is *totally geodesic* if

$$2\langle\nabla_Y Z, X\rangle := \langle[X,Y],Z\rangle + \langle[X,Z],Y\rangle = 0, \text{ for all } X\in\mathfrak{h}^\perp, Y,Z\in\mathfrak{h}. \tag{1}$$

The definition is chosen so that the corresponding Lie subgroup is totally geodesic in the usual sense.

This paper is the sequel to [1], in which we gave a number of results concerning totally geodesic subalgebras of nilpotent Lie algebras. In particular, we showed that in each dimension $n\geq 3$, there is up to isomorphism only one filiform nilpotent Lie algebra that possesses a totally geodesic subalgebra of codimension two. We showed that in filiform nilpotent Lie algebras, totally geodesic subalgebras that leave invariant their orthogonal complements have dimension at most half the dimension of the algebra. And we gave an example of a 6-dimensional filiform nilpotent Lie algebra that has no totally geodesic subalgebras of dimension >2, for any choice of inner product.

In this paper, we focus on an important natural family of nilpotent Lie algebras.

Definition 1. *We say an n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is* $\mathbb{N}$-graded filiform, *if it can be decomposed in a direct sum of one dimensional subspaces $\mathfrak{g}=\bigoplus_{i=1}^n V_i$ with $[V_1,V_i]=V_{i+1}$ for all $i>1$ and $[V_i,V_j]\subset V_{i+j}$ for all $i,j\in\mathbb{N}$, where for convenience we set $V_i=0$ for $i>n$.*

These algebras have been completely classified by Millionshchikov [5]. There are 6 natural sequences of algebras of arbitrary large dimension, and in addition, in each dimension from 7 through to 11, there is a one parameter family of exceptional algebras. The purpose of our paper is to establish the following result:

Theorem 1. *Suppose that $\mathfrak{h}$ is a subalgebra of an $\mathbb{N}$-graded filiform Lie algebra $\mathfrak{g}$ and that $\mathfrak{h}$ is totally geodesic with respect to some inner product on $\mathfrak{g}$. Then one of the following conditions holds:*

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(a) $\mathfrak{g}$ *has a presentation with basis* $X_1, \dots, X_n$ *and relations* $[X_1, X_i] = X_{i+1}$ *for all* $1 < i < n$. *In this case* $\dim(\mathfrak{h}) \leq \dim(\mathfrak{g}) - 2$.
(b) $\mathfrak{g}$ *has a presentation with basis* $X_1, \dots, X_{2k+1}$ *and relations* $[X_1, X_i] = X_{i+1}$ *for all* $1 < i < 2k+1$ *and* $[X_l, X_{2k+1-l}] = (-1)^{l+1} X_{2k+1}$ *for all* $1 < l < k+1$. *In this case* $\dim(\mathfrak{h}) \leq \dim(\mathfrak{g}) - 4$.
(c) $\mathfrak{g}$ *is not isomorphic to one of the algebras in parts* (a) *and* (b), *in which case* $\dim(\mathfrak{h}) \leq \lfloor \dim(\mathfrak{g})/2 \rfloor$.

Moreover, each of the above bounds is attained for some inner product and subalgebra $\mathfrak{h}$.

The paper is organised as follows. In the next section we recall some background information, including Millionshchikov's classification. In Section 3 we establish some general preliminary results. Section 4 deals with the 6 natural families of $\mathbb{N}$-graded filiform Lie algebras. Finally, Section 5 treats the exceptional algebras in dimension 7 through to 11.

2. Background

In 1983, Fialowski classified all infinite-dimensional $\mathbb{N}$-graded filiform Lie algebras, and later some of her results were rediscovered by Khakimdjanova and Khakimdjanov in [4].

Theorem 2 ([2]). *Let* $\mathfrak{g}$ *be an infinite-dimensional* $\mathbb{N}$*-graded filiform Lie algebra. Then* $\mathfrak{g}$ *is isomorphic to precisely one of the following Lie algebras:*
(a) $\mathfrak{m}_0 = \mathrm{Span}(X_i, i \in \mathbb{N} \mid [X_1, X_j] = X_{j+1}, \forall j \geq 2)$,
(b) $\mathfrak{m}_2 = \mathrm{Span}(X_i, i \in \mathbb{N} \mid [X_1, X_j] = X_{j+1}, \forall j \geq 2,\ [X_2, X_j] = X_{j+2}, \forall j \geq 3)$,
(c) $\mathcal{V} = \mathrm{Span}(X_i, i \in \mathbb{N} \mid [X_i, X_j] = (j-i) X_{i+j}, \forall i, j)$.

If $\mathfrak{g} = \bigoplus_{i=1}^{\infty} V_i$ is an infinite-dimensional $\mathbb{N}$-graded filiform Lie algebra, then the quotient algebra $\mathfrak{g}(n) = \mathfrak{g} / \bigoplus_{i=n+1}^{\infty} V_i$ is an n-dimensional $\mathbb{N}$-graded filiform Lie algebra. Therefore, in this way we can obtain three natural sequences of finite-dimensional algebras, that we denote: $\mathfrak{m}_0(n)$, $\mathfrak{m}_2(n)$, $\mathcal{V}_n$. In 1991, Khakimdjanov proved in [3] that there only exists a finite number of non-isomorphic $\mathbb{N}$-graded filiform Lie algebras over $\mathbb{C}$ in dimensions ≥ 12. Then, in 2004, Millionshchikov classified all finite-dimensional $\mathbb{N}$-graded filiform Lie algebras over an arbitrary field $\mathbb{K}$ of characteristic zero [5].

Theorem 3 ([5]). *Let* $\mathfrak{g}$ *be a finite-dimensional* $\mathbb{N}$*-graded filiform Lie algebra. Then* $\mathfrak{g}$ *is isomorphic to a Lie algebra from the following list:*
(a) *the six sequences* $\mathfrak{m}_0(n)$, $\mathfrak{m}_2(n)$, $\mathcal{V}_n$, $\mathfrak{m}_{0,1}(2k+1)$, $\mathfrak{m}_{0,2}(2k+2)$, $\mathfrak{m}_{0,3}(2k+3)$, *defined by the basis* $\{X_1, \dots, X_n\}$ *and commutation relations given in Table 1;*
(b) *the 5 one-parameter families* $\mathfrak{g}_{n,\alpha}$ *of dimensions* $n = 7, \dots, 11$ *respectively, defined by their basis and commutation relations given in Table 2.*

Remark 1. For $\alpha = -2$ we have $\mathfrak{g}_{7,-2} \cong \mathfrak{m}_{0,1}(7)$, $\mathfrak{g}_{8,-2} \cong \mathfrak{m}_{0,2}(8)$ and $\mathfrak{g}_{9,-2} \cong \mathfrak{m}_{0,3}(9)$. These isomorphisms seem to have been overlooked in [5]. Apart from these, the algebras in Tables 1 and 2 are pair-wise non-isomorphic. However, if we drop the restrictions on dimensions of the algebras in Table 1, then we also have the following isomorphisms: $\mathfrak{m}_0(3) \cong \mathfrak{m}_2(3) \cong \mathcal{V}_3$, $\mathfrak{m}_0(4) \cong \mathfrak{m}_2(4) \cong \mathcal{V}_4$, $\mathfrak{m}_2(5) \cong \mathcal{V}_5$, $\mathfrak{m}_2(6) \cong \mathcal{V}_6$, and for $\alpha = 8$ we have $\mathfrak{g}_{n,8} \cong \mathcal{V}_n$ where $n = 7, \dots, 11$, as one can see by examining the basis $\{X_1, \frac{1}{(k-2)! \cdot 60} X_k : k = 2, \dots, n\}$.

algebra	dimension	presentation
$\mathfrak{m}_0(n)$	$n \geq 3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, n-1$
$\mathfrak{m}_2(n)$	$n \geq 5$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, n-1$ $[X_2, X_i] = X_{i+2}, \quad i = 3, \dots, n-2$
$\mathcal{V}_n$	$n \geq 12$	$[X_i, X_j] = \begin{cases} (j-i)X_{i+j}, & i+j \leq n; \\ 0, & i+j > n; \end{cases}$
$\mathfrak{m}_{0,1}(2k+1)$	$n = 2k+1$, $k \geq 3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k$
$\mathfrak{m}_{0,2}(2k+2)$	$n = 2k+2$, $k \geq 3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k+1$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k$ $[X_j, X_{2k-j+2}] = (-1)^{j+1}(k-j+1) X_{2k+2}, \quad j = 2, \dots, k$
$\mathfrak{m}_{0,3}(2k+3)$	$n = 2k+3$, $k \geq 3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k+2$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k$ $[X_j, X_{2k-j+2}] = (-1)^{j+1}(k-j+1) X_{2k+2}, \quad j = 2, \dots, k$ $[X_m, X_{2k-m+3}] = (-1)^m((m-2)k - \frac{(m-2)(m-1)}{2}) X_{2k+3}, \quad m = 3, \dots, k+1$

Table 1. Six infinite sequences of $\mathbb{N}$-graded filiform Lie algebra

3. Preliminaries

Throughout this paper, $\mathfrak{g}$ is an $\mathbb{N}$-graded filiform Lie algebra of dimension n. We fix the decomposition $\mathfrak{g} = \bigoplus_{i=1}^n V_i$ with $[V_1, V_i] = V_{i+1}$ for all $i > 1$ and $[V_i, V_j] \subset V_{i+j}$ for all $i, j \in \mathbb{N}$, where for convenience, we set $V_i = 0$ for $i > n$. Introduce the ideals $\mathfrak{g}_i := \oplus_{j \geq i} V_j$, for $i = 1, \dots, n$. For $Y \in \mathfrak{g}$, we define the *degree* of Y, denoted $\deg(Y)$, to be the largest natural number k such that $Y \in \mathfrak{g}_k$, and for convenience, we set $\deg(0) = \infty$.

We choose a basis $\mathcal{B} = \{X_1, \dots, X_n\}$ for $\mathfrak{g}$ with $X_i \in V_i$ for $i = 1, \dots, n$. Obviously, $\deg([X_i, X_j]) \geq i + j$, for all i, j. Consequently, if $Y_1, Y_2 \in \mathfrak{g}$ such that $\deg(Y_1) = i$ and $\deg(Y_2) = j$, then $\deg([Y_1, Y_2]) \geq i+j$. Moreover, if $\deg([X_i, X_j]) = i+j$, then $\deg([Y_1, Y_2]) = i + j$.

Remark 2. If we take an inner product on $\mathfrak{g}$ for which $X_1, \dots, X_n$ are orthonormal, then the subalgebra $\mathfrak{h}$ generated by $\{X_i : i \text{ is even}\}$ is a totally geodesic subalgebra of dimension $\lfloor n/2 \rfloor$.

Now assume $\mathfrak{g}$ is equipped with an arbitrary inner product $\langle \cdot, \cdot \rangle$. Applying the Gram-Schmidt orthonormalisation procedure to $\mathcal{B}$, starting with the element of the largest degree, we obtain an orthonormal basis $\mathcal{E} = \{E_1, \dots, E_n\}$, where for each i one has $\deg(E_i) = i$. Clearly, $\mathrm{Span}(E_k, \dots, E_n) = \mathfrak{g}_k$. Note that by construction, $[E_i, E_j] \in \mathfrak{g}_{i+j}$ for all i, j.

algebra	restrictions	presentation
$\mathfrak{g}_{7,\alpha}$	$\alpha \neq -2$	$[X_1, X_j] = X_{j+1}, \quad 2 \leq j \leq 6$ $[X_2, X_3] = (2+\alpha)X_5,\ [X_2, X_4] = (2+\alpha)X_6,$ $[X_2, X_5] = (1+\alpha)X_7, \quad [X_3, X_4] = X_7,$
$\mathfrak{g}_{8,\alpha}$	$\alpha \neq -2$	relations of $\mathfrak{g}_{7,\alpha}$ and: $[X_1, X_7] = X_8,\ [X_2, X_6] = \alpha X_8, \quad [X_3, X_5] = X_8,$
$\mathfrak{g}_{9,\alpha}$	$\alpha \neq -\frac{5}{2}, -2$	relations of $\mathfrak{g}_{8,\alpha}$ and: $[X_1, X_8] = X_9, \quad [X_2, X_7] = \frac{2\alpha^2+3\alpha-2}{2\alpha+5}X_9,$ $[X_3, X_6] = \frac{2\alpha+2}{2\alpha+5}X_9, \quad [X_4, X_5] = \frac{3}{2\alpha+5}X_9,$
$\mathfrak{g}_{10,\alpha}$	$\alpha \neq -\frac{5}{2}$	relations of $\mathfrak{g}_{9,\alpha}$ and: $[X_1, X_9] = X_{10}, \quad [X_2, X_8] = \frac{2\alpha^2+\alpha-1}{2\alpha+5}X_{10},$ $[X_3, X_7] = \frac{2\alpha-1}{2\alpha+5}X_{10}, \quad [X_4, X_6] = \frac{3}{2\alpha+5}X_{10},$
$\mathfrak{g}_{11,\alpha}$	$\alpha \neq -\frac{5}{2}, -1, -3$	relations of $\mathfrak{g}_{10,\alpha}$ and: $[X_1, X_{10}] = X_{11}, \quad [X_2, X_9] = \frac{2\alpha^3+2\alpha^2+3}{2(\alpha^2+4\alpha+3)}X_{11},$ $[X_3, X_8] = \frac{4\alpha^3+8\alpha^2-8\alpha-21}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11},$ $[X_4, X_7] = \frac{3(2\alpha^2+4\alpha+5)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11},$ $[X_5, X_6] = \frac{3(4\alpha+1)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11}$

TABLE 2. Five one-parameter families of $\mathbb{N}$-graded filiform Lie algebra

Furthermore, $\langle [E_1, E_i], E_{i+1} \rangle \neq 0$ for all $1 < i < n$. So E_1 has maximal nilpotency. For more details see [1].

Now let $\mathfrak{h}$ be a totally geodesic subalgebra of $\mathfrak{g}$ of dimension greater than 1. We will repeatedly use the following facts:

Lemma 1. *We have:*

(a) $E_1 \in \mathfrak{h}^\perp$ *and* $\mathfrak{h} \subset \mathrm{Span}(X_2, \dots, X_n)$.
(b) *It is impossible that* $E_i, E_{i+1} \in \mathfrak{h}$*, for any* $i = 2, \dots, n-1$*. In particular, there is an element of degree 2 or 3 in* $\mathfrak{h}^\perp$.
(c) *If* $E_i + aE_{i+1} \in \mathfrak{h}$*, for any* $i = 2, \dots, n-1$*, then* $a = 0$.
(d) *Suppose* $E_n, Y \in \mathfrak{h}$ *where* $\deg(Y) = k$ *for some* $k = 2, \dots, n-1$*. If* $[X_k, X_{n-k}] = aX_n$ $(a \neq 0)$*, then there is no* $Z \in \mathfrak{h}^\perp$ *such that* $\deg(Z) = n-k$.
(e) *Suppose* $E_{n-1}, Y \in \mathfrak{h}$ *where* $\deg(Y) = k$*, for some* $k = 2, \dots, n-3$*. If* $[X_k, X_{n-k-1}] = bX_{n-1}$ $(b \neq 0)$*, then there is no* $Z \in \mathfrak{h}^\perp$ *such that* $\deg(Z) = n-k-1$.

Proof. (a) By [1, Lemma 4.4], the elements of degree one have maximal nilpotency. Then by [1, Lemma 4.6], $\mathfrak{h} \subset \mathfrak{g}_2$. Hence $E_1 \in \mathfrak{h}^\perp$; see [1, Remark 4.7]. So $\mathfrak{h} \subset \mathrm{Span}(E_2, \dots, E_n) = \mathrm{Span}(X_2, \dots, X_n)$.

(b) Suppose that $E_i, E_{i+1} \in \mathfrak{h}$, for some $i = 2, \dots, n-1$. As $E_1 \in \mathfrak{h}^\perp$ then

$$2\langle \nabla_{E_i} E_{i+1}, E_1 \rangle = \langle [E_1, E_i], E_{i+1} \rangle + \langle [E_1, E_{i+1}], E_i \rangle = \langle [E_1, E_i], E_{i+1} \rangle \neq 0,$$

which would contradict (1). Therefore, there exists an element $Z \in \mathfrak{h}^\perp$ of degree 2 or 3, since otherwise $E_2, E_3 \in \mathfrak{h}$.

(c) From (1) we have $0 = \langle [E_1, E_i + aE_{i+1}], E_i + aE_{i+1} \rangle = a\langle [E_1, E_i], E_{i+1} \rangle$, so $a = 0$.

(d) By (1), if $Y, E_n \in \mathfrak{h}$, $\deg(Y) = k$ and there is $Z \in \mathfrak{h}^\perp$ such that $\deg(Z) = n - k$, then we have

$$0 = \langle [Z, Y], E_n \rangle + \langle [Z, E_n], Y \rangle = \langle [Z, Y], E_n \rangle.$$

However, since $\deg([X_k, X_{n-k}]) = n$, we have $\deg([Z, Y]) = n$, which is a contradiction.

(e) By (1), if $Y, E_{n-1} \in \mathfrak{h}$, $\deg(Y) = k$ and there is $Z \in \mathfrak{h}^\perp$ such that $\deg(Z) = n - k - 1$, then we have

$$0 = \langle [Z, Y], E_{n-1} \rangle + \langle [Z, E_{n-1}], Y \rangle = \langle [Z, Y], E_{n-1} \rangle.$$

But since $\deg([X_k, X_{n-k-1}]) = n - 1$, we have $\deg([Z, Y]) = n - 1$, which is a contradiction. □

Lemma 2. *Suppose that $X_n \notin \mathfrak{h}$ and there are elements of degree $p, \dots, n-1$ in $\mathfrak{h}$, for some $p \geq 2$. Then there are no elements of degree $p+2, \dots, n$ in $\mathfrak{h}^\perp$.*

Proof. If $X_n \notin \mathfrak{h}$ and there are elements $Y_i \in \mathfrak{h}$ with $\deg(Y_i) = i$ for $i = p, \dots, n-1$, then by taking linear combinations if necessary, we may take $Y_i := E_i + a_i E_n$ for some $a_i \in \mathbb{R}$. Note that $a_{n-1} = 0$ by Lemma 1(c), so $E_{n-1} \in \mathfrak{h}$. Moreover $a_{n-2} \neq 0$, as otherwise we would have $E_{n-2}, E_{n-1} \in \mathfrak{h}$, contradicting Lemma 1(b). So there are no elements of degree $n-1$ or n in $\mathfrak{h}^\perp$.

The rest of the proof is done by induction. Assume that for some k with $p + 2 < k < n$, there are no elements of degree $k, \dots, n$ in $\mathfrak{h}^\perp$, but there is some $Z_{k-1} \in \mathfrak{h}^\perp$ with $\deg(Z_{k-1}) = k - 1$. Since Z_{k-1} and Y_{k-1} are orthogonal, we have $a_{k-1} \neq 0$ and $\langle Z_{k-1}, E_n \rangle \neq 0$. Then from the orthogonality of $\mathfrak{h}^\perp$ and $\mathfrak{h}$ we obtain $a_{k-2}, a_{k-3} = 0$ giving $E_{k-2}, E_{k-3} \in \mathfrak{h}$, which is impossible, by Lemma 1(b). □

Let $\mathcal{O}_1$ be the family of n-dimensional $\mathbb{N}$-graded filiform Lie algebras for which the basis $\mathcal{B} = \{X_1, \dots, X_n\}$ may be chosen so that

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, \quad i = 2, \dots, n-1; \\ [X_i, X_{n-i}] &= \alpha_i X_n, \quad (\alpha_i \neq 0), \quad i = 2, \dots, \lfloor (n-1)/2 \rfloor. \end{aligned} \tag{2}$$

Additionally, denote by $\mathcal{O}_2$ the subfamily of $\mathcal{O}_1$ comprised of algebras for which $\mathcal{B}$ may be chosen so that conditions (2) and condition

$$[X_2, X_i] = \beta_i X_{i+2}, \quad (\beta_i \neq 0) \quad i = 3, \dots, n-2 \tag{3}$$

are satisfied.

Lemma 3. *Suppose that $X_n \notin \mathfrak{h}$ and that $n \geq 5$.*

(a) *If $\mathfrak{g}$ belongs to $\mathcal{O}_1$, then* $\dim(\mathfrak{h}) \leq \lfloor n/2 \rfloor$*,*
(b) *If $\mathfrak{g}$ belongs to $\mathcal{O}_2$, then* $\dim(\mathfrak{h}) \leq \lfloor (n-1)/2 \rfloor$*.*

Proof. (a) Suppose $\mathfrak{g}$ belongs to $\mathcal{O}_1$. Since $X_n \notin \mathfrak{h}$, there is no elements of $\mathfrak{h}$ of degree n and if there is an element of $\mathfrak{h}$ of degree k, then as $\mathfrak{h}$ is a subalgebra, there are no elements of $\mathfrak{h}$ of degree $n-k$ except if n is even and $k = \frac{n}{2}$. Therefore, if n is odd, $\dim(\mathfrak{h}) \le \frac{n-1}{2}$, and if n is even, $\dim(\mathfrak{h}) \le \frac{n}{2}$. This proves (a).

(b) Suppose $\mathfrak{g}$ belongs to $\mathcal{O}_2$. From the proof of (a), we may assume that n is even and that $\mathfrak{h}$ contains an element $Y_{\frac{n}{2}}$ of degree $\frac{n}{2}$. First suppose there is an element $Y_2 \in \mathfrak{h}$ with $\deg(Y_2) = 2$. Let $m := \frac{n}{2}$. If m is even, then $U = \mathrm{ad}^{\frac{m}{2}}(Y_2)(Y_m) \in \mathfrak{h}$ and $\deg(U) = n$ contradicting the assumption. If m is odd and $n \ne 6$, then since $\mathfrak{h}$ would contain either an element Y_{m-1} of degree $m-1$ or an element Y_{m+1} of degree $m+1$, we would obtain either $V_1 = \mathrm{ad}^{\frac{m-1}{2}+1}(Y_2)(Y_{m-1}) \in \mathfrak{h}$ and $\deg(V_1) = n$ or $V_2 = \mathrm{ad}^{\frac{m-1}{2}}(Y_2)(Y_{m+1}) \in \mathfrak{h}$ and $\deg(V_2) = n$, respectively. In each case, we get a contradiction with the assumption. Let us discuss the case $n = 6$. By Lemma 1(a), we have $E_1 \in \mathfrak{h}^\perp$. Then from the above argument, we may assume that $\mathfrak{h} = \mathrm{Span}(Y_2, Y_3, Y_5)$ for some Y_i such that $\deg(Y_i) = i$. By Lemma 1(c), we have $\langle E_6, Y_5\rangle = 0$. If there is $Z_2 \in \mathfrak{h}^\perp$ with $\deg(Z_2) = 2$, then

$$2\langle \nabla_{Y_3} Y_5, Z_2\rangle = \langle [Z_2, Y_3], Y_5\rangle \ne 0,$$

which is impossible by (1). If there is $Z_3 \in \mathfrak{h}^\perp$ with $\deg(Z_3) = 3$, then

$$0 = 2\langle \nabla_{Y_2} Y_5, Z_3\rangle = \langle [Z_3, Y_2], Y_5\rangle,$$

is also impossible. The remaining case is $\mathfrak{h} = \mathrm{Span}(E_2, E_3, E_5)$, which is impossible by Lemma 1(b).

If there are no elements of degree 2 in $\mathfrak{h}$ then $E_2 \in \mathfrak{h}^\perp$ and in order to be a totally geodesic subalgebra of dimension $\frac{n}{2}$, $\mathfrak{h}$ has to contain an element of degree $n-1$ as well as an element of degree $n-2$, say Y_{n-1} and Y_{n-2} respectively. By (1), conditions

$$\langle \nabla_{Y_{n-1}} Y_{n-1}, E_1\rangle = 0 \quad \text{and} \quad \langle \nabla_{Y_{n-2}} Y_{n-2}, E_2\rangle = 0$$

give

$$\langle X_n, Y_{n-1}\rangle = 0 \quad \text{and} \quad \langle X_n, Y_{n-2}\rangle = 0.$$

But then $E_{n-1}, E_{n-2} \in \mathfrak{h}$, contradicting Lemma 1(b). $\square$

Lemma 4. *Let $\mathfrak{g}$ be a Lie algebra with an inner product $\langle\cdot,\cdot\rangle$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a totally geodesic subalgebra.*

(a) *If $\mathfrak{g}$ is nilpotent, then the projection of its center to $\mathfrak{h}$ lies in the center of $\mathfrak{h}$.*

(b) *Let $\mathfrak{i} \subset \mathfrak{g}$ be an ideal. Consider the Lie algebra $\overline{\mathfrak{g}} := \mathfrak{g}/\mathfrak{i}$ and the quotient map $\pi : \mathfrak{g} \to \overline{\mathfrak{g}}$. Suppose that $\mathfrak{i} = (\mathfrak{i}\cap\mathfrak{h})\oplus(\mathfrak{i}\cap\mathfrak{h}^\perp)$. Then there is an inner product on $\overline{\mathfrak{g}}$ for which $\overline{\mathfrak{h}} := \pi(\mathfrak{h})$ is a totally geodesic subalgebra.*

Proof. (a) Let Z be a vector from the center of $\mathfrak{g}$ and let $Z = Z_{\mathfrak{h}} + Z_\perp$, $Z_{\mathfrak{h}} \in \mathfrak{h}, Z_\perp \in \mathfrak{h}^\perp$. By (1), $\langle [Z_\perp, X], X\rangle = 0$, for all $X \in \mathfrak{h}$, hence $\langle [Z_{\mathfrak{h}}, X], X\rangle = 0$, for all $X \in \mathfrak{h}$. As $\mathfrak{h}$ is a subalgebra, it is an invariant subspace of the operator $\mathrm{ad}(Z_{\mathfrak{h}})$. Moreover, the restriction of $\mathrm{ad}(Z_{\mathfrak{h}})$ to $\mathfrak{h}$ is both nilpotent and skew-symmetric, hence is zero, so $[Z_{\mathfrak{h}}, \mathfrak{h}] = 0$.

(b) Let $\mathfrak{h}'$ and $\mathfrak{h}'_\perp$ be the orthogonal complements to $\mathfrak{i}$ in $\mathfrak{h}$ and in $\mathfrak{h}^\perp$ respectively and let $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{h}'_\perp$. We equip the linear space $\mathfrak{g}'$ with the inner product induced from that on $\mathfrak{g}$ and with a bilinear skew-symmetric map $[\cdot,\cdot]'$ defined by $[X, Y]' = \pi'([X, Y])$, where $\pi' : \mathfrak{g} \to \mathfrak{g}'$ is the orthogonal projection. As $\mathfrak{i}$ is an ideal, $[\cdot,\cdot]'$ is a Lie bracket, which turns $\mathfrak{g}'$ into a Lie algebra isomorphic to $\overline{\mathfrak{g}}$, with $\mathfrak{h}' \subset \mathfrak{g}'$ a subalgebra isomorphic to $\overline{\mathfrak{h}}$. Moreover, as $\pi'(\mathfrak{h}) = \mathfrak{h}'$

and $\pi'(\mathfrak{h}^\perp) = \mathfrak{h}'_\perp$, the validity of condition (1) for the vectors from $\mathfrak{h}'$ and $\mathfrak{h}'_\perp$ follows from that for the vectors from $\mathfrak{h}$ and $\mathfrak{h}^\perp$. $\square$

Lemma 5. *Suppose that $X_n \in \mathfrak{h}$ and that $n \geq 6$.*

(a) *If n is even and $\mathfrak{g}/\operatorname{Span}(X_n) \in \mathcal{O}_1$, then $\dim(\mathfrak{h}) \leq n/2$,*
(b) *If n is odd and $\mathfrak{g}/\operatorname{Span}(X_n) \in \mathcal{O}_2$, then $\dim(\mathfrak{h}) \leq (n-1)/2$.*

Proof. Consider the quotient map $\pi : \mathfrak{g} \to \overline{\mathfrak{g}} := \mathfrak{g}/\operatorname{Span}(X_n)$. By Lemma 4(b), there is an inner product on $\overline{\mathfrak{g}}$ for which $\overline{\mathfrak{h}} := \pi(\mathfrak{h})$ is a totally geodesic subalgebra. Let $\overline{X}_i := \pi(X_i)$ for all $i = 1, \ldots, n-1$. Note that $\overline{X}_{n-1} \notin \overline{\mathfrak{h}}$ since otherwise we would have $X_{n-1}, X_n \in \mathfrak{h}$, contradicting Lemma 1(b). Hence if n is even and $\overline{\mathfrak{g}} \in \mathcal{O}_1$, then $\dim(\overline{\mathfrak{h}}) \leq \lfloor (n-1)/2 \rfloor = (n-2)/2$ by Lemma 3(a), so $\dim(\mathfrak{h}) \leq n/2$. Similarly, if n is odd and $\overline{\mathfrak{g}} \in \mathcal{O}_2$, then $\dim(\overline{\mathfrak{h}}) \leq \lfloor (n-3)/2 \rfloor = (n-3)/2$ by Lemma 3(b), so $\dim(\mathfrak{h}) \leq (n-1)/2$. $\square$

4. Algebras from Table 1

We treat the algebras in the order they appear in Table 1. First recall that by [1, Theorem 1.17], for all $n \geq 3$, the Lie algebra $\mathfrak{m}_0(n)$ possesses an inner product relative to which $\mathfrak{m}_0(n)$ has a totally geodesic subalgebra of codimension two. This result is optimal since by [1, Proposition 1.13], filiform Lie algebras have no totally geodesic subalgebras of codimension one.

Remark 3. With the exception of $\mathfrak{m}_0(n)$, the algebras of Tables 1 and 2 have no totally geodesic subalgebras of codimension one, by [1, Proposition 1.13], and none of codimension two, by [1, Theorem 1.18].

Theorem 4. *If $\mathfrak{h}$ is a proper totally geodesic subalgebra of $\mathfrak{m}_2(n)$, then $\dim(\mathfrak{h}) \leq \frac{n}{2}$.*

Proof. Suppose that $\mathfrak{m}_2(n)$ has an inner product $\langle \cdot, \cdot \rangle$ for which $\mathfrak{h}$ is a totally geodesic subalgebra and assume that $\dim(\mathfrak{h}) > \frac{n}{2}$. First assume that there are no elements in $\mathfrak{h}$ of degree two. So there exist elements of $\mathfrak{h}^\perp$ of the following form: $Z = X_1 + \sum_{i=3}^n a_i X_i$ and $W = X_2 + \sum_{i=3}^n b_i X_i$, for some $a_i, b_i \in \mathbb{R}$. Note that $\mathfrak{h}$ is contained in the derived algebra $[\mathfrak{m}_2(n), \mathfrak{m}_2(n)]$ of $\mathfrak{m}_2(n)$. The key observation is that when restricted to $[\mathfrak{m}_2(n), \mathfrak{m}_2(n)]$, one has $\operatorname{ad}(W) = \operatorname{ad}^2(Z)$.

Let $\mathfrak{k} = \mathfrak{h}^\perp \cap [\mathfrak{m}_2(n), \mathfrak{m}_2(n)]$ and consider the map $f : \mathfrak{h} \to \mathfrak{k}$ defined by $f(Y) = \pi_\perp([Z, Y])$, where $\pi_\perp : \mathfrak{m}_2(n) \to \mathfrak{k}$ is the orthogonal projection. We have $\dim(\mathfrak{h}) > \frac{n}{2} = \frac{\dim(\mathfrak{h}) + \dim(\mathfrak{k}) + 2}{2}$, so $\dim(\mathfrak{h}) > \dim(\mathfrak{k}) + 2$. Thus $\dim(\ker(f)) > 2$. Since $\dim(\ker(\operatorname{ad}(Z)_{|\mathfrak{h}})) = 1$, there exists $Y \in \mathfrak{h}$ with $Y \in \ker(f) \setminus \ker(\operatorname{ad}(Z))$; that is, $[Z, Y] \in \mathfrak{h}$ and $[Z, Y] \neq 0$. Since $\mathfrak{h}$ is totally geodesic, $Z \in \mathfrak{h}^\perp$ and $Y, [Z, Y] \in \mathfrak{h}$, we have

$$0 = \langle Y, [Z, [Z, Y]] \rangle + \langle [Z, Y], [Z, Y] \rangle$$

so $\langle Y, [Z, [Z, Y]] \rangle \neq 0$. But $\langle Y, [Z, [Z, Y]] \rangle = \langle Y, [W, Y] \rangle = 0$, since $\mathfrak{h}$ is totally geodesic, $W \in \mathfrak{h}^\perp$ and $Y \in \mathfrak{h}$. This is a contradiction.

It remains to consider the case where there exists $Y_2 \in \mathfrak{h}$ with $\deg(Y_2) = 2$. First suppose that $E_n \notin \mathfrak{h}$. If n is odd then, since $\mathfrak{h}$ has an element of degree 2, $\mathfrak{h}$ can have no elements of odd degree and so $\dim(\mathfrak{h}) \leq \frac{n-1}{2}$. If n is even then $\mathfrak{h}$ can have no elements of even degree ≥ 4, and consequently $\dim(\mathfrak{h}) \leq \frac{n}{2}$.

Now suppose that $E_n \in \mathfrak{h}$. Note that by Lemma 1(b), $\mathfrak{h}$ has no elements of degree $n-1$. Hence, if n is even, $\mathfrak{h}$ can have no elements of odd degree, and so $\dim(\mathfrak{h}) \leq \frac{n}{2}$. If n is odd, $\mathfrak{h}$ can have no elements of even degree ≥ 4, and consequently $\dim(\mathfrak{h}) \leq \frac{n+1}{2}$. Suppose therefore that n is odd and that $\mathfrak{h}$ has dimension $\frac{n+1}{2}$; so there are elements Y_i for $i = 3, 5, 7, \dots, n$ such that Y_i has degree i and $\mathfrak{h} = \operatorname{Span}(Y_2, Y_3, Y_5, \dots, Y_n)$. Without loss of generality, we may assume that for each $i = 3, 5, 7, \dots, n$, the vector Y_i has no components in the E_j direction for all odd $j > i$.

Note that $\mathfrak{h}^\perp$ has no elements of degree 2. Indeed, if W were such an element, then $[W, Y_{n-2}]$ would have degree n, but we would also have

$$0 = \langle Y_{n-2}, [W, Y_n]\rangle + \langle [W, Y_{n-2}], Y_n\rangle = \langle [W, Y_{n-2}], Y_n\rangle,$$

which is impossible.

We claim that $\mathfrak{h}^\perp$ has no elements of odd degree ≥ 3. Indeed, suppose that $\mathfrak{h}^\perp$ has an element W of odd degree $i \geq 3$. Note that $[W, Y_2]$ has degree $i+2$ and for each odd $j \geq 3$, we have $Y_j, W \in \operatorname{Span}(X_3, X_4, \dots, X_n)$ and so $[W, Y_j] = 0$. So

$$0 = \langle Y_2, [W, Y_j]\rangle + \langle [W, Y_2], Y_j\rangle = \langle [W, Y_2], Y_j\rangle.$$

Moreover, $\langle Y_2, [W, Y_2]\rangle = 0$. Hence $[W, Y_2] \in \mathfrak{h}^\perp$. By induction, we obtain an element of $\mathfrak{h}^\perp$ of degree n, contradicting the assumption that $E_n \in \mathfrak{h}$.

From what we have just seen, since $\dim(\mathfrak{h}^\perp) = \frac{n-1}{2}$, there are necessarily elements W_i for $i = 4, 6, 8, \dots, n-1$ such that W_i has degree i and $\mathfrak{h}^\perp = \operatorname{Span}(E_1, W_4, W_6, \dots, W_{n-1})$. We may assume that for each $i = 4, 6, 8, \dots, n-1$, the vector W_i has no components in the E_j direction for all even $j > i$. Note that as W_{n-1} has degree $n-1$, we have $W_{n-1} \in \operatorname{Span}(E_{n-1}, E_n)$. So, as W_{n-1} is perpendicular to Y_n, which is a multiple of E_n, we must have that W_{n-1} is a multiple of E_{n-1}. Continuing by induction, it is clear that W_i is a multiple of E_i for all $i = 4, 6, 8, \dots, n-1$, and Y_i is a multiple of E_i for all $i = 2, 3, 5, 7, \dots, n$. But then as $Y_2, Y_3 \in \mathfrak{h}$ we have $E_2, E_3 \in \mathfrak{h}$, which contradicts Lemma 1(b). This completes the proof of the theorem. $\square$

Theorem 5. *If $\mathfrak{h}$ is a proper totally geodesic subalgebra of $\mathcal{V}_n$, $n \geq 3$, then $\dim(\mathfrak{h}) \leq \frac{n}{2}$.*

Proof. By [1, Proposition 1.13], filiform Lie algebras have no totally geodesic subalgebras of codimension one. So the result is true for $n \leq 4$. For $n = 5$ the claim follows from Theorem 4 and the isomorphism from Remark 1. Suppose $n \geq 6$. By Lemma 3 we may assume that $X_n \in \mathfrak{h}$ and then the required result follows immediately from Lemma 5, as $\mathcal{V}_n / \operatorname{Span}(X_n) \cong \mathcal{V}_{n-1} \in \mathcal{O}_2$. $\square$

Theorem 6. *The minimal codimension of a proper totally geodesic subalgebra $\mathfrak{h}$ of $\mathfrak{m}_{0,1}(2k+1)$, $k \geq 3$, is four.*

Proof. Let $\mathfrak{h}$ be a proper totally geodesic subalgebra of $\mathfrak{m}_{0,1}(2k+1)$, $k \geq 3$. If $X_{2k+1} \notin \mathfrak{h}$, then Lemma 3(a) gives $\dim(\mathfrak{h}) \leq k$, since $\mathfrak{m}_{0,1}(2k+1) \in \mathcal{O}_1$ for every $k \geq 3$. So we may assume that $X_{2k+1} \in \mathfrak{h}$. Hence, by Lemma 1(b), there are no elements of degree $2k$ in $\mathfrak{h}$. By Lemma 1(a), $\mathfrak{h} \subset \operatorname{Span}(X_2, \dots, X_{2k+1})$. Moreover, by [1, Proposition 1.13, Theorem 1.18], the codimension of $\mathfrak{h}$ is at least 3. If the codimension of $\mathfrak{h}$ is 3, there exists $2 \leq i \leq 2k-1$ such that $\mathfrak{h} = \operatorname{Span}(Y_2, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_{2k-1}, X_{2k+1})$, where $\deg(Y_j) = j$. We may then choose a basis for the subspace $\mathfrak{g}' = \mathfrak{h} \oplus \operatorname{Span}(X_i) \subset \mathfrak{g}$ of the form $\{X_j + a_j X_{2k}, X_{2k+1}\}$, $j =$

$2, \dots, 2k-1$, for some $a_j \in \mathbb{R}$, hence $\mathfrak{g}'$ is a Lie algebra isomorphic to the Heisenberg algebra. It follows from (1) that $\mathfrak{h}$ is totally geodesic in $\mathfrak{g}'$, with the induced inner product, but this is a contradiction with [1, Proposition 1.13]. Hence $\mathfrak{h}$ has codimension at least 4 in $\mathfrak{m}_{0,1}(2k+1)$.

In the following, we will exhibit an example of a totally geodesic subalgebra $\mathfrak{h}$ of $\mathfrak{m}_{0,1}(2k+1)$ of codimension exactly four. Fix $k \geq 3$ and let $\mathfrak{m} = \mathbb{R}^{2k+1}$, equipped with an inner product $\langle \cdot, \cdot \rangle$ and an orthonormal basis $\{E_1, \dots, E_{2k+1}\}$. Introduce the subspace $\mathfrak{m}' = \mathrm{Span}(E_2, \dots, E_{2k+1})$, and define a bilinear skew-symmetric map $[\cdot, \cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$[E_1, X] = NX, \quad [X, Y] = \langle KX, Y \rangle E_{2k+1}, \quad \text{for all } X, Y \in \mathfrak{m}', \tag{4}$$

where the operators $N, K \in \mathrm{End}(\mathfrak{m}')$ are defined by their matrices relative to the basis $E_2, \dots, E_{2k+1}$ for $\mathfrak{m}'$ as follows:

$$N = \left(\begin{array}{cc|cc} 0 & S & 0 & 0 \\ -S + uu^t & 0 & 0 & 0 \\ \hline p^t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \qquad K = \left(\begin{array}{cc|cc} 0 & I_{k-1} & 0 & 0 \\ -I_{k-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \tag{5}$$

where I_{k-1} is the identity matrix, $u, p \in \mathbb{R}^{k-1}$, and S is a symmetric nonsingular $(k-1) \times (k-1)$-matrix such that the matrix

$$T = S(-S + uu^t) \text{ is nilpotent.} \tag{6}$$

We postpone the question of the existence of such S and u and of a correct choice of p to a little later.

The map $[\cdot, \cdot]$ given by (4) can be extended to $\mathfrak{m}$ by skew symmetry and bilinearity (note that K is skew-symmetric). The claim of the theorem is established in the following four steps:

(i) The space $\mathfrak{m}$ with the map $[\cdot, \cdot]$ defined by (4, 5) is a Lie algebra.
(ii) The subspace $\mathfrak{h} = \big(\mathrm{Span}(E_1, E_{2k}, (0, u, 0_{k+1})^t, (0_k, u, 0, 0)^t) \big)^\perp$ is a totally geodesic subalgebra of $\mathfrak{m}$, where 0_m denotes the row vector of m zeros.
(iii) There exists $u \in \mathbb{R}^{k-1}$ and a symmetric nonsingular matrix S satisfying (6).
(iv) There exists $p \in \mathbb{R}^{k-1}$ such that the Lie algebra $\mathfrak{m}$ defined by (4, 5, 6), with S and u constructed as in (iii), is isomorphic to $\mathfrak{m}_{0,1}(2k+1)$.

(i). To see this, it suffices to check the Jacobi identities. As $[\mathfrak{m}, E_{2k+1}] = 0$ and $[\mathfrak{m}', \mathfrak{m}'] = \mathrm{Span}(E_{2k+1})$, they are satisfied for any triple of vectors from $\mathfrak{m}'$. By (4), the Jacobi identity on a triple (E_1, X, Y), $X, Y \in \mathfrak{m}'$, is equivalent to $\langle KNX, Y \rangle = \langle KNY, X \rangle$, for all $X, Y \in \mathfrak{m}'$, which is true, as KN is symmetric (from (5)).

(ii). Note that by (6), $u \neq 0$ as S is nonsingular, so $\mathrm{codim}(\mathfrak{h}) = 4$. The subspace $\mathfrak{h}$ is a subalgebra since $[\mathfrak{h}, \mathfrak{h}] \subset [\mathfrak{m}', \mathfrak{m}'] \subset \mathrm{Span}(E_{2k+1}) \subset \mathfrak{h}$. To see that $\mathfrak{h}$ is totally geodesic, we have to check that (1) is satisfied. If $X \in \mathfrak{m}'$, equation (1) is equivalent to $\langle KX, Y \rangle \langle E_{2k+1}, Z \rangle + \langle KX, Z \rangle \langle E_{2k+1}, Y \rangle = 0$. As K is skew-symmetric, this is equivalent to $\langle X, \langle E_{2k+1}, Z \rangle KY + \langle E_{2k+1}, Y \rangle KZ \rangle = 0$, which is true since $KY, KZ \in \mathfrak{h}$, as $\mathfrak{h} \subset \mathfrak{m}'$ is K-invariant (see (4)).

If $X = E_1$, then by (4), condition (1) is equivalent to $\langle NY, Z \rangle + \langle NZ, Y \rangle = 0$, for all $Y, Z \in \mathfrak{h}$, which follows from the form of N given in (4) and the definition of $\mathfrak{h}$.

(iii). Suppose that a nonsingular symmetric operator S and a vector u satisfy

$$\begin{gathered}\operatorname{rk}(S^{-1}u, S^{-3}u, \dots, S^{3-2k}u) = k-1, \\ \langle S^{-1}u, u\rangle = 1, \quad \langle S^{-3}u, u\rangle = \dots = \langle S^{3-2k}u, u\rangle = 0.\end{gathered} \tag{7}$$

Then by the second condition of (7), $TS^{-1}u = S(-S+uu^t)S^{-1}u = 0$, and then $T^2S^{-3}u = TS(-S+uu^t)S^{-3}u = -TS^{-1}u = 0$, and, by induction, $T^jS^{1-2j}u = 0$, for all $j = 1, \dots, k-1$. So

$$T^mS^{1-2j}u = 0, \text{ for all } 1 \le j \le m \le k-1, \tag{8}$$

and in particular, $T^{k-1}S^{-1}u = T^{k-1}S^{-3}u = \dots = T^{k-1}S^{3-2k}u = 0$. As by the first condition of (7) the vectors $S^{-1}u, \dots, S^{3-2k}u$ form a basis for $\mathbb{R}^{k-1}$, we obtain $T^{k-1} = 0$, as required.

To construct S and u satisfying (7), consider a diagonal matrix S with distinct positive diagonal entries d_i, for $i = 1, \dots, k-1$, and a vector $u = (u_1, \dots, u_{k-1})^t$ none of whose entries are zero. The first condition of (7) is equivalent to the condition that the $(k-1) \times (k-1)$ matrix M with entries $M_{ij} = d_i^{1-2j}u_i$ is nonsingular. As $\det(M) = \prod_i(d_i^{-1}u_i) \times V(d_1^{-2}, \dots, d_{k-1}^{-2})$, where V is the Vandermonde determinant, the first condition is satisfied since $d_i^2 \neq d_j^2$ for $i \neq j$. The second condition of (7) is equivalent to the following condition:

$$\sum_{i=1}^{k-1} d_i^{1-2l}u_i^2 = \begin{cases} 1 & : \text{ if } l = 1, \\ 0 & : \text{ otherwise,} \end{cases} \tag{9}$$

for $l = 1, \dots, k-1$. To obtain this, we choose the u_i's and adjust the signs of the d_i's in such a way that $d_i^{-1}u_i^2 = \prod_{j\neq i} d_i^2(d_i^2 - d_j^2)^{-1}$, for $i = 1, \dots, k-1$. To see that this works, we employ the following combinatorial result:

Lemma 6. *Let $b_1, \dots, b_m$ be distinct nonzero reals, where $m \geq 2$. Then for $0 \le l \le m-1$,*

$$\sum_{i=1}^{m} b_i^l \prod_{j\neq i} \frac{1}{b_i - b_j} = \begin{cases} 1 & : \textit{ if } l = m-1, \\ 0 & : \textit{ otherwise.} \end{cases}$$

Proof. Let $\phi_i(t) = \prod_{j\neq i}(t - b_j^{-1})(b_i^{-1} - b_j^{-1})^{-1}$. So $\phi_i(b_j^{-1}) = \delta_{ij}$ for all $1 \le i, j \le m$. Consider the polynomial $f_l(t) = \sum_{i=1}^m b_i^{l-m+1}\phi_i(t)$. Note that $f_l(t)$ has degree $m-1$ and $f_l(b_i^{-1}) = b_i^{-(m-l-1)}$ for each i. So $f_l(t) = t^{m-l-1}$. So

$$\begin{aligned}\sum_{i=1}^{m} b_i^l \prod_{j\neq i} \frac{1}{b_i - b_j} &= \sum_{i=1}^{m} b_i^{l-m+1} \prod_{j\neq i} \frac{b_i}{b_i - b_j} \\ &= \sum_{i=1}^{m} b_i^{l-m+1} \prod_{j\neq i} \frac{-b_j^{-1}}{b_i^{-1} - b_j^{-1}} = f_l(0) = \delta_{l,m-1}.\end{aligned}$$

□

Setting $m = k-1$ and $b_i = d_i^2$, the lemma gives

$$\sum_{i=1}^{k-1} d_i^{1-2l} u_i^2 = \sum_{i=1}^{k-1} d_i^{2-2l} \prod_{j\neq i} d_i^2 (d_i^2 - d_j^2)^{-1} = \sum_{i=1}^{k-1} d_i^{2k-2-2l} \prod_{j\neq i} (d_i^2 - d_j^2)^{-1}$$
$$= \sum_{i=1}^{m} b_i^{m-l} \prod_{j\neq i} \frac{1}{b_i - b_j} = \delta_{1l},$$

which gives (9) as required. This completes (iii).

(iv). We first consider the requirement that $\mathfrak{m}$ be filiform, with E_1 having maximal nilpotency. Note that $\mathfrak{m}'$ contains the derived algebra of $\mathfrak{m}$ and is isomorphic to the direct product of the Heisenberg algebra $Heis(2k-1)$ and the one-dimensional abelian ideal spanned by E_{2k} (and hence is nilpotent). It therefore suffices to show that the restriction of $\mathrm{ad}(E_1)$ to $\mathfrak{m}'$ is nilpotent and has a maximal rank, that is, that N is a nilpotent matrix and that $N^{2k-1} \neq 0$. For $m \geq 1$, we have by induction:

$$N^{2m+1} = \left(\begin{array}{cc|cc} 0 & S(T^t)^m & 0 & 0 \\ (-S+uu^t)T^m & 0 & 0 & 0 \\ \hline p^t T^m & 0 & 0 & 0 \\ 0 & p^t S (T^t)^{m-1} & 0 & 0 \end{array}\right). \tag{10}$$

The $(k-1)\times(k-1)$ matrix T is nilpotent by (6), so $T^{k-1} = 0$, so N (and hence $\mathfrak{m}$) is indeed nilpotent. Moreover, for $m = k-1$ we have

$$N^{2k-1} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & (T^{k-2}Sp)^t & 0 & 0 \end{array}\right), \tag{11}$$

so $\mathfrak{m}$ is filiform if and only if

$$T^{k-2}Sp \neq 0. \tag{12}$$

Let us assume for the moment that (12) is satisfied. To show that $\mathfrak{m}$ is isomorphic to $\mathfrak{m}_{0,1}(2k+1)$, we have to find a vector $X_2 \in \mathfrak{m}'$ such that the vectors $X_i = N^{i-2}X_2$, $i = 2, \dots, 2k+1$, are nonzero and satisfy $[X_i, X_j] = (-1)^{i+1}\delta_{i+j,2k+1}X_{2k+1}$, for all $i, j = 2, \dots, 2k+1$. We choose the vector X_2 whose coordinates relative to the basis $E_2, \dots, E_{2k+1}$ for $\mathfrak{m}'$ are $(X_2)^t = (0_{k-1}, q^t, 0, 0)$, where $q \in \mathbb{R}^{k-1}$ satisfies

$$\langle T^{k-2}Sp, q\rangle = -1. \tag{13}$$

Then $X_{2k+1} = N^{2k-1}X_2 = -E_{2k+1}$ by (11), so by (4), the vector q has to be chosen in such a way that $\langle KN^{i-2}X_2, N^{j-2}X_2\rangle = (-1)^i\delta_{i+j,2k+1}$, for all $i, j = 2, \dots, 2k+1$. Note that the matrix KN is symmetric, so $N^tK = -KN$. It follows that if $i+j$ is even (say $i+j = 2s$), then $\langle KN^{i-2}X_2, N^{j-2}X_2\rangle = \pm\langle KN^{s-2}X_2, N^{s-2}X_2\rangle = 0$, as K is skew-symmetric. If $i+j = 2s+1$ is odd, we have $\langle KN^{i-2}X_2, N^{j-2}X_2\rangle = (-1)^{j-2}\langle KN^{2s-3}X_2, X_2\rangle = (-1)^{i+1}\langle KN^{2s-3}X_2, X_2\rangle$. So we require $\langle KN^{2s-3}X_2, X_2\rangle = -\delta_{sk}$, for all $s = 2, \dots, k$. By (10) and the choice of X_2, the latter equation is equivalent to $\langle S(T^t)^m q, q\rangle = \delta_{m,k-2}$, for all $m = 0, \dots, k-2$ (note that $T^{k-1} = 0$ anyway). As $ST^t = TS$, this is equivalent to

$$\langle T^m Sq, q\rangle = \delta_{m,k-2}, \text{ for all } m = 0, \dots, k-2. \tag{14}$$

Define $q = S^{2-2k}P(S^2)u$, where $P(t) = a_0 + a_1 t + \dots + a_{k-2}t^{k-2}$ is a polynomial which we will specify a little later. By (8), $T^m S^{1-2i}u = 0$, for $1 \le i \le m \le k-1$. Moreover, since $T = S(-S + uu^t)$ one obtains by induction using (7) that $T^m S^{1-2i}u = (-1)^m S^{2m+1-2i}u$, for $0 \le m < i \le k-1$. It follows that $\langle T^m S^{1-2i}u, S^{-2j}u\rangle = 0$, when $1 \le i \le m \le k-1$, and $\langle T^m S^{1-2i}u, S^{-2j}u\rangle = (-1)^m \langle S^{2m+1-2i-2j}u, u\rangle$, when $0 \le m < i \le k-1$, so, again by (7), $\langle T^m S^{1-2i}u, S^{-2j}u\rangle = 0$, when $i + j < k + m$, and $\langle T^m S^{1-2i}u, S^{-2j}u\rangle = (-1)^m \langle S^{2m+1-2i-2j}u, u\rangle$, if $k + m \le i + j \le 2(k-1)$. It follows that

$$
\begin{aligned}
\langle T^m Sq, q\rangle &= \sum\nolimits_{s,l=0}^{k-2} a_s a_l \langle T^m S^{3-2k+2s}u, S^{2-2k+2l}u\rangle \\
&= (-1)^m \sum\nolimits_{s,l;0\le s+l\le k-m-2} a_s a_l \langle S^{2m+5-4k+2s+2l}u, u\rangle \\
&= (-1)^m \sum\nolimits_{M=0}^{k-m-2} \langle S^{2m+5-4k+2M}u, u\rangle \big(\sum\nolimits_{s,l\ge 0; s+l=M} a_s a_l\big) \\
&= (-1)^m \sum\nolimits_{M=0}^{k-2} \langle S^{2m+5-4k+2M}u, u\rangle [P^2]_M,
\end{aligned}
$$

where $[P^2]_M$ is the coefficient of t^M in the polynomial $P^2(t)$, and where we changed the upper limit in the second summation from $k-2$ to $k-m-2$, since for $m \le k-2$ and $k-m-1 \le M \le k-2$, we have $3 - 2k \le 2m + 5 - 4k + 2M \le 2m + 1 - 2k \le -3$, so $\langle S^{2m+5-4k+2M}u, u\rangle = 0$ by (7). It follows that

$$\langle T^m Sq, q\rangle = (-1)^m \langle S^{2m+5-4k} \sum\nolimits_{M=0}^{k-2} [P^2]_M S^{2M}u, u\rangle, \tag{15}$$

To choose the polynomial P, we need the following lemma.

Lemma 7. *Let $\chi(t)$ be a polynomial of degree r with a positive constant term. Then there exists a polynomial $P(t)$ of degree r, such that $t^{r+1} \mid (P^2(t) - \chi(t))$.*

Proof. Let $\chi(t) = a^2 + b_1 t + \dots + b_r t^r$, $a \ne 0$. Formally, $P(t)$ is just the truncation of the formal power series $a(1 + (a^{-2}b_1 t + \dots + b_r t^r))^{1/2}$ up to the term t^r. Informally, for $P(t) = c_0 + c_1 t + \dots + c_r t^r$, we have $c_0 = a$, and then $2c_0c_1 = b_1$, which can be solved for c_1; then $2c_0c_2 + c_1^2 = b_2$, which can be solved for c_2, and so on. $\square$

Now in Lemma 7, take $r = k-2$ and $\chi(t) = \det(S^2 - tI_{k-1}) + (-1)^k t^{k-1}$ (note that $\deg(\chi) = k-2$), and choose the corresponding $P(t)$. Then $\sum_{M=0}^{k-2}[P^2]_M S^{2M} = \sum_{M=0}^{k-2}[\chi]_M S^{2M} = \chi(S^2) = (-1)^k S^{2k-2}$, by the Cayley–Hamilton theorem. Then from (15) and from (7) we obtain that for all $m = 0, \dots, k-2$,

$$\langle T^m Sq, q\rangle = (-1)^{m+k} \langle S^{2m+3-2k}u, u\rangle = (-1)^{m+k}\delta_{m,k-2} = \delta_{m,k-2},$$

as required by (14). As by (14), the vector $(T^m S)^t q$ is clearly nonzero, we can find $p \in \mathbb{R}^{k-1}$ satisfying (13), and then (12) will be satisfied automatically.

This proves the last step, and hence, completes the proof of the theorem. $\square$

Theorem 7. *The maximal dimension of a proper totally geodesic subalgebra $\mathfrak{h}$ of $\mathfrak{m}_{0,2}(2k+2)$, $k \ge 3$, is $k + 1$.*

Proof. As $\mathfrak{m}_{0,2}(2k+2) \in \mathcal{O}_1$, we have $\dim(\mathfrak{h}) \leq k+1$ by Lemma 3(a), if $X_{2k+2} \notin \mathfrak{h}$. So we may suppose that $X_{2k+2} \in \mathfrak{h}$, in which case the claim follows from Lemma 5(a), as $\mathfrak{m}_{0,2}(2k+2)/\operatorname{Span}(X_n) \cong \mathfrak{m}_{0,1}(2k+1) \in \mathcal{O}_1$. □

Theorem 8. *The maximal dimension of a proper totally geodesic subalgebra $\mathfrak{h}$ of $\mathfrak{m}_{0,3}(2k+3)$, $k \geq 3$, is $k+1$.*

Proof. Let $\mathfrak{g} = \mathfrak{m}_{0,3}(2k+3)$, $k \geq 3$, with the inner product $\langle \cdot, \cdot \rangle$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a proper totally geodesic subalgebra. First note that if $X_{2k+3} \in \mathfrak{h}$, then we have the quotient map

$$\pi : \mathfrak{g} \to \overline{\mathfrak{g}} := \mathfrak{m}_{0,3}(2k+3)/\operatorname{Span}(X_{2k+3}) \cong \mathfrak{m}_{0,2}(2k+2).$$

and by Lemma 4(b), there is an inner product on $\overline{\mathfrak{g}}$ for which $\overline{\mathfrak{h}} := \pi(\mathfrak{h})$ is a totally geodesic subalgebra. Then by Theorem 7, we have $\dim(\overline{\mathfrak{h}}) \leq k+1$ and so $\dim(\mathfrak{h}) \leq k+2$. On the other hand, if $X_{2k+3} \notin \mathfrak{h}$, then for all $m = 3, \dots, k+1$, since $[X_m, X_{2k-m+3}]$ is a nonzero multiple of X_{2k+3}, the subalgebra $\mathfrak{h}$ cannot have an element of degree m and an element of degree $2k-m+3$. Thus, using Lemma 1(a), we again have $\dim(\mathfrak{h}) \leq k+2$. So in either case, we have $\dim(\mathfrak{h}) \leq k+2$, while we require $\dim(\mathfrak{h}) \leq k+1$.

Consider the subalgebra $\mathfrak{g}' = \operatorname{Span}(X_2, \dots, X_{2k+3})$. Note that $\mathfrak{g}'$ is two-step nilpotent and that $\mathfrak{n} := [\mathfrak{g}', \mathfrak{g}'] = \operatorname{Span}(X_{2k+1}, X_{2k+2}, X_{2k+3})$ is its center. Let $\mathfrak{b}$ denote the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}'$; so $\mathfrak{b}$ is a $(2k-1)$-dimensional vector subspace of $\mathfrak{g}'$, but not a subalgebra. For $N \in \mathfrak{n}$, define a skew-symmetric operator $J_N \in \mathfrak{so}(\mathfrak{b})$ by $\langle J_N X, Y\rangle = \langle N, [X,Y]\rangle$, for $X, Y \in \mathfrak{b}$. We have the following lemma, which roughly says that although the two-step algebra $\mathfrak{g}'$ is not non-degenerate, its degeneracy can be tightly controlled.

Lemma 8.

(a) *If $N \in \mathfrak{n}$ is nonzero, then* $\operatorname{rk}(J_N) = 2k-2$.
(b) *If $N \in \mathfrak{n}$ is nonzero and $V \subset \mathfrak{b}$ is a subspace of maximal dimension such that $\langle J_N V, V\rangle = 0$, then $\dim(V) = k$ and $\ker(J_N) \subset V$.*
(c) *If $N_1, N_2 \in \mathfrak{n}$ are not proportional, then* $\ker(J_{N_1}) \cap \ker(J_{N_2}) = 0$.
(d) *If $N_1, N_2 \in \mathfrak{n}$ are not proportional, then the space $L = \operatorname{Span}(\ker(J_N) \mid N \in \operatorname{Span}(N_1, N_2), N \neq 0)$ has dimension k.*

Proof. The subspace $\mathfrak{b}$ has a basis (possibly non-orthonormal) of the form $Y_i = X_i + Z_i$, $i = 2, \dots, 2k$, where $Z_i \in \mathfrak{n}$. Note that $[Y_i, Y_j] = [X_i, X_j]$, for all $i, j = 2, \dots, 2k$. The matrix of the operator J_N relative to the orthonormal basis E_i, $i = 2, \dots, 2k$, for $\mathfrak{b}$ is given by $J_N = \sum_{\alpha=1}^{3} \langle X_{2k+\alpha}, N\rangle B K_\alpha B^t$, where B is the transformation matrix between the bases Y_i and E_i, and the skew-symmetric matrices K_α, $\alpha = 1, 2, 3$, are given by the defining relations of $\mathfrak{m}_{0,3}(2k+3)$, relative to the basis X_i (that is, $[X_i, X_j] = \sum_{\alpha=1}^{3} (K_\alpha)_{i-1,j-1} X_{2k+\alpha}$, for $i, j = 2, \dots, 2k$). Explicitly, they are given by

$$\begin{gathered}(K_1)_{lm} = (-1)^l \delta_{l+m,2k-1}, \quad (K_2)_{lm} = (-1)^l \delta_{l+m,2k}(k-l), \\ (K_3)_{lm} = (-1)^{l+1} \delta_{l+m,2k+1} \frac{1}{2}(l-1)(m-1),\end{gathered} \tag{16}$$

for $l, m = 1, \dots, 2k-1$. We have $J_N = B(aK_1 + bK_2 + cK_3)B^t$, where $a = \langle X_{2k+1}, N\rangle$, $b = \langle X_{2k+2}, N\rangle$, $c = \langle X_{2k+3}, N\rangle$. As B is nonsingular, it suffices to prove all four assertions of

the lemma, with the J_N's replaced by the matrices $aK_1+bK_2+cK_3$. From (16),

$$aK_1+bK_2+cK_3= \tag{17}$$

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a & -(k-1)b \\ 0 & 0 & 0 & \dots & a & (k-2)b & -(k-1)c \\ 0 & 0 & 0 & \iddots & -(k-3)b & (2k-3)c & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 0 & -a & (k-3)b & \iddots & 0 & 0 & 0 \\ a & -(k-2)b & -(2k-3)c & \dots & 0 & 0 & 0 \\ (k-1)b & (k-1)c & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

(a) As $aK_1+bK_2+cK_3$ is skew-symmetric and as $\dim(\mathfrak{b})=2k-1$ is odd, $\dim(\ker(aK_1+bK_2+cK_3))$ is odd. Assume that it is of dimension at least three. Suppose that $c\neq 0$. Then there exists a nonzero vector $x=(x_1,x_2,\dots,x_{2k-1})^t\in\ker(aK_1+bK_2+cK_3)$ whose first two coordinates are zero: $x_1=x_2=0$. Multiplying the matrix given by (17) by such an x we obtain $x_3=0$ from the second last row, then $x_4=0$ from the third last row, and so on (note that all the entries on the sub-antidiagonal are nonzero by (16)). Then $x=0$, a contradiction. A similar argument (starting with the last two coordinates being zero) also works when $a\neq 0$. If $a=c=0$, $b\neq 0$, then from (17), $\ker(K_2)$ is spanned by the k-th coordinate vector.

(b) As the subspaces V and J_NV are orthogonal and as $\dim(\ker(J_N))=1$ by assertion (a), we have $2k-1=\dim(\mathfrak{b})\geq\dim(V)+\dim(J_NV)\geq 2\dim(V)-1$. It follows that $\dim(V)\leq k$, with the equality only possible when $\dim(V)=k$ and $\dim(J_NV)=k-1$, which implies $\ker(J_N)\subset V$.

(c) By assertion (a), the kernel of any nonzero $K=aK_1+bK_2+cK_3$ has dimension one. We can find that kernel explicitly, namely

$$\begin{gathered}\ker(aK_1+bK_2+cK_3)=\operatorname{Span}(x),\ \text{where } x=(x_1,\dots,x_{2k-1})^t,\\ \sum\nolimits_{j=0}^{2k-2}(j!)^{-1}x_{2k-1-j}t^j=(a-bt+\tfrac{1}{2}ct^2)^{k-1}.\end{gathered} \tag{18}$$

To prove this, assume that $x=(x_1,\dots,x_{2k-1})^t$ is a nonzero vector from $\ker(aK_1+bK_2+cK_3)$ and define $y_j=x_{2k-1-j}$ for $j=0,\dots,2k-2$. Then from (16) we obtain $ay_{j+1}+b(k-j-1)y_j-\frac{1}{2}cj(2k-j-1)y_{j-1}=0$, for all $j=0,\dots,2k-2$ (with the understanding that $y_{-1}=y_{2k-1}=0$). Multiplying by $t^j/j!$ and summing up by $j=0,\dots,2k-2$ we obtain $ap'(t)+b(-tp'(t)+(k-1)p(t))+c(\frac{1}{2}t^2p'(t)-(k-1)tp)=0$, where $p(t)=\sum_{j=0}^{2k-2}y_jt^j/j!$. It follows that the polynomial $p(t)$ satisfies the equation $(a-tb+\frac{1}{2}ct^2)p'(t)=(k-1)(a-tb+\frac{1}{2}ct^2)'p(t)$, a solution to which is $(a-bt+\frac{1}{2}ct^2)^{k-1}$. It follows from (18) that two nonzero matrices $a_1K_1+b_1K_2+c_1K_3$ and $a_2K_1+b_2K_2+c_2K_3$ have the same kernel only when the polynomials $(a_i-b_it+\frac{1}{2}c_it^2)^{k-1}$, $i=1,2$, are proportional, which implies $(a_1,b_1,c_1)\parallel(a_2,b_2,c_2)$.

(d) From (18), the dimension of the span of the kernels of all the nontrivial linear combinations of nonproportional matrices $a_1K_1+b_1K_2+c_1K_3$ and $a_2K_1+b_2K_2+c_2K_3$ is the same as that of the subspace $S=\operatorname{Span}_{\lambda\in\mathbb{R}}(((a_1+\lambda a_2)-(b_1+\lambda b_2)t+\frac{1}{2}(c_1+\lambda c_2)t^2)^{k-1})$

in the space of polynomials in t of degree less than or equal to $2k-2$. The subspace S is spanned by the polynomials $(a_1 - b_1 t + \frac{1}{2}c_1 t^2)^{k-1-j}(a_2 - b_2 t + \frac{1}{2}c_2 t^2)^j$, $j = 0, \dots, k-1$. If they were linearly dependent, then there would exist a nontrivial relation of the form $\sum_{j=0}^{k-1} \mu_j (a_1 - b_1 t + \frac{1}{2}c_1 t^2)^{k-1-j}(a_2 - b_2 t + \frac{1}{2}c_2 t^2)^j = 0$. Dividing both sides by $(a_1 - b_1 t + \frac{1}{2}c_1 t^2)^{k-1}$ we obtain that a nonconstant rational function $f(t) = (a_1 - b_1 t + \frac{1}{2}c_1 t^2)^{-1}(a_2 - b_2 t + \frac{1}{2}c_2 t^2)$ satisfies a nontrivial polynomial equation $\sum_{j=0}^{k-1} \mu_j (f(t))^j = 0$ for all $t \in \mathbb{R}$ (for which $f(t)$ is defined), a contradiction. $\square$

It follows from Lemma 1(a) that $\mathfrak{h} \subset \mathfrak{g}'$. Let $d = \dim(\mathfrak{h} \cap \mathfrak{n})$. We have $d \le \dim(\mathfrak{n}) = 3$. First of all, note that d cannot equal 3. Indeed, otherwise we would have $X_{2k+2}, X_{2k+3} \in \mathfrak{h}$, contradicting Lemma 1(b). Therefore $d \le 2$.

Let $N \in \mathfrak{n}$ be a unit vector orthogonal to $\mathfrak{h} \cap \mathfrak{n}$. As $\mathfrak{h}$ is a subalgebra and as $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{n}$, we have $\langle [\mathfrak{h}, \mathfrak{h}], N \rangle = 0$. As $\mathfrak{n}$ is in the center of $\mathfrak{g}'$, we obtain $\langle J_N V, V \rangle = 0$, where $V = \pi_{\mathfrak{b}}(\mathfrak{h})$ and $\pi_{\mathfrak{b}} \in \operatorname{End}(\mathfrak{g}')$ is the orthogonal projection to $\mathfrak{b}$. It follows that $\dim(\mathfrak{h}) = \dim(\pi_{\mathfrak{b}}(\mathfrak{h})) + \dim(\ker(\pi_{\mathfrak{b}}) \cap \mathfrak{h}) = \dim(V) + d \le k+2$, from Lemma 8(b), with the equality achieved only when $\dim(V) = k$ and $d = 2$.

So if $\dim(\mathfrak{h}) = k+2$, then $d = 2$ and $\mathfrak{h} = \operatorname{Span}(v_1 + \lambda_1 N, \dots, v_k + \lambda_k N, N_2, N_3)$, where the vectors $v_i \in \mathfrak{b}$ are linearly independent, $\operatorname{Span}_{i=1}^k (v_i) = V$, and $\{N, N_2, N_3\}$ is an orthonormal basis for $\mathfrak{n}$. Then $\mathfrak{h}^\perp = \operatorname{Span}(E_1, w_1 + \mu_1 N, \dots, w_k + \mu_k N)$, where E_1 is the unit vector orthogonal to $\mathfrak{g}'$, and $w_j \in \mathfrak{b}$ for $j = 1, \dots, k$. Denote $W = \operatorname{Span}_{i=1}^k (w_i) = \pi_{\mathfrak{b}}(\mathfrak{h}^\perp)$. Although the vectors w_j may not be linearly independent, we have $V + W = \mathfrak{b}$, so $\dim(W) \ge k-1$. We have $\dim(W) \le k$, so $\dim(W)$ is either k or $k-1$.

Now by equation (1), with $X = w_j + \mu_j N$, $Y = v_i + \lambda_i N$ and $Z = N_\alpha$, $\alpha = 2, 3$, we obtain $\langle [w_j, v_i], N_\alpha \rangle = 0$, so $\langle J_{N_\alpha} V, W \rangle = 0$, for $\alpha = 2, 3$.

If $\dim(W) = k$, then $\dim(J_{N_\alpha} V) \le k-1$, so $\ker(J_{N_\alpha}) \cap V \ne 0$. By Lemma 8(a), $\dim(\ker(J_{N_\alpha})) = 1$, and so $\ker(J_{N_\alpha}) \subset V$. By the same reasoning, $\ker(J_{N_\alpha}) \subset W$ and so $\ker(J_{N_\alpha}) \subset V \cap W$. Since $\dim(\mathfrak{b}) = 2k-1$ and $\dim(V) = \dim(W) = k$, we have $\dim(V \cap W) = 1$. Hence $\ker(J_{N_\alpha}) = V \cap W$, for $\alpha = 2, 3$. But this contradicts Lemma 8(c).

Next suppose that $\dim(W) = k-1$. Specifying the basis for $\mathfrak{h}^\perp$ we can assume that $w_k = 0$ and that $w_1, \dots, w_{k-1} \in \mathfrak{b}$ are linearly independent, so $\mathfrak{h}^\perp = \operatorname{Span}(E_1, w_1, \dots, w_{k-1}, N)$. Then $\langle w_j, v_i \rangle = 0$, so W is the orthogonal complement to V in $\mathfrak{b}$. It now follows from $\langle J_{N_\alpha} V, W \rangle = 0$, for $\alpha = 2, 3$, that V and W are complementary invariant subspaces of both J_{N_2} and J_{N_3}, hence of any J_Z, where Z is a nonzero linear combination of N_2 and N_3. By Lemma 8(a), $\dim(\ker(J_Z)) = 1$. As the projections of the kernel to invariant subspaces again lie in the kernel, we obtain that $\ker(J_Z)$ is a subspace of either V or W. By continuity, the union $\mathcal{U}$ of the kernels of J_Z, taken over all nonzero $Z \in \operatorname{Span}(N_2, N_3)$, lies either in V or in W. But by Lemma 8(d), $\dim(\operatorname{Span}(\mathcal{U})) = k$, so $\operatorname{Span}(\mathcal{U}) = V$ (as $\dim(V) = k$ and $\dim(W) = k-1$). Now take any two nonproportional $X, Y \in \mathcal{U}$. There exist nonzero vectors $Z_1 = aN_2 + bN_3$, $Z_2 = cN_2 + dN_3$ such that $\ker(J_{Z_1}) = \operatorname{Span}(X)$, $\ker(J_{Z_2}) = \operatorname{Span}(Y)$ and moreover, by Lemma 8(a), Z_1 and Z_2 are nonproportional, so $\operatorname{Span}(Z_1, Z_2) = \operatorname{Span}(N_2, N_3)$. As $\langle J_{Z_1} X, Y \rangle = \langle J_{Z_2} X, Y \rangle = 0$, we obtain that $\langle J_{N_2} X, Y \rangle = \langle J_{N_3} X, Y \rangle = 0$, for any pair of nonproportional vectors $X, Y \in \mathcal{U}$, hence, for arbitrary any pair of vectors $X, Y \in \mathcal{U}$, hence, for arbitrary vectors $X, Y \in V = \operatorname{Span}(\mathcal{U})$. It follows that $\langle J_{N_\alpha} V, V \rangle = 0$, for $\alpha = 2, 3$. As from the above, $\langle J_{N_\alpha} V, W \rangle = 0$, where W is the orthogonal complement to V

in $\mathfrak{b}$, we obtain that $\ker(J_{N_\alpha})$ contains the k-dimensional space V, which strongly contradicts Lemma 8(a). □

5. Algebras from Table 2

We treat the algebras $\mathfrak{g}_{n,\alpha}$ in the order they appear in Table 2. In this section there are numerous cases and subcases. For convenience, we adopt the following notational convention throughout this section: the expression Y_i denotes an element of $\mathfrak{h}$ of degree i. Similarly, the expression Z_i denotes an element of $\mathfrak{h}^\perp$ of degree i. Furthermore, when we choose a basis $Y_{i_1}, \dots, Y_{i_k}$, $i_1 < i_2 < \cdots < i_k$, for $\mathfrak{h}$, the assumption is that the elements are chosen so that $\langle Y_{i_j}, E_{i_j} \rangle = 1$ for each j, and that Y_{i_j} has no component in the direction E_{i_l} for $l > j$. However we do not impose similar restrictions on the coefficients of our bases for $\mathfrak{h}^\perp$.

Theorem 9. *Let $\mathfrak{h}$ be a proper totally geodesic subalgebra of $\mathfrak{g}_{7,\alpha}$. Then, for all $\alpha \in \mathbb{R}\backslash\{-2\}$, we have* $\dim(\mathfrak{h}) \leq 3$.

Proof. By Remark 3, our goal is to show that $\mathfrak{h}$ cannot have dimension 4. First suppose that $X_7 \in \mathfrak{h}$. As $\mathfrak{g}_{7,\alpha}/\operatorname{Span}(X_7) \cong \mathfrak{m}_2(6) \in \mathcal{O}_2$, we have $\dim(\mathfrak{h}) \leq 3$ by Lemma 5(b). Hence, we may suppose that $X_7 \notin \mathfrak{h}$.

Note that if $\alpha \neq -1$, then $\mathfrak{g}_{7,\alpha} \in \mathcal{O}_1$, and Lemma 3(a) gives the required result. So it remains to treat the case $\alpha = -1$. Suppose that $\mathfrak{h}$ has dimension 4. Since $[X_3, X_4] = X_7 \notin \mathfrak{h}$ and $E_1 \in \mathfrak{h}^\perp$ by Lemma 1(a), we can choose a basis so that $\mathfrak{h} = \operatorname{Span}(Y_2, Y_i, Y_5, Y_6)$, where $i = 3$ or 4. Note that by Lemma 1(c), $Y_6 = E_6$. Then we have $Y_5 = E_5 + a_5 E_7$, for some $a_5 \in \mathbb{R}$, where $a_5 \neq 0$ by Lemma 1(b). So we may assume that Y_2, Y_i have no components in the E_5 and E_6 directions. Moreover, the projection of E_7 to $\mathfrak{h}$ is nonzero, as $a_5 \neq 0$ and has degree at least 5, as it lies in the center of $\mathfrak{h}$ by Lemma 4(a) and as $[X_2, X_3]$ and $[X_2, X_4]$ are both nonzero. It follows that $\pi_\mathfrak{h}(E_7) = \frac{a_5}{1+a_5^2} Y_5 = \frac{a_5}{1+a_5^2} E_5 + \frac{a_5^2}{1+a_5^2} E_7$, and so the vector $Z_5 = (1 + a_5^2)(\pi_\mathfrak{h}(E_7) - E_7) = a_5 E_5 - E_7$ belongs to $\mathfrak{h}^\perp$. Then both Y_2 and Y_i have no component in the E_7 direction.

Now if $i = 3$, we have $Y_3 = E_3$ by Lemma 1(c) and we may take $Y_2 = E_2 + cE_4$, where $c \neq 0$ by Lemma 1(b). But then $Z_2 = E_4 - cE_2 \in \mathfrak{h}^\perp$ and we get a contradiction with (1), as $[Z_2, E_6] = 0$, but $\langle [Z_2, Y_2], E_6 \rangle = -(1 + c^2)\langle [E_2, E_4], E_6 \rangle \neq 0$.

If $i = 4$, then $Y_4 = E_4$ and we may take $Y_2 = E_2 + bE_3$, so $Y_2 = E_2$ by Lemma 1(c). Then $E_3 \in \mathfrak{h}^\perp$ and we get a contradiction with (1), as $[Y_5, E_3] = 0$, but $\langle [E_3, E_4], Y_5 \rangle = a_5 \langle [E_3, E_4], E_7 \rangle \neq 0$. □

Theorem 10. *Let $\mathfrak{h}$ be a proper totally geodesic subalgebra of $\mathfrak{g}_{8,\alpha}$. Then, for all $\alpha \in \mathbb{R}\backslash\{-2\}$, we have* $\dim(\mathfrak{h}) \leq 4$.

Proof. First suppose that $X_8 \in \mathfrak{h}$. Note that $\mathfrak{g}_{8,\alpha}/\operatorname{Span}(X_8) \cong \mathfrak{g}_{7,\alpha}$. Consider the quotient map $\pi : \mathfrak{g}_{8,\alpha} \to \mathfrak{g}_{7,\alpha}$. By Lemma 4(b), there is an inner product on $\mathfrak{g}_{7,\alpha}$ for which $\bar{\mathfrak{h}} := \pi(\mathfrak{h})$ is a totally geodesic subalgebra. Thus by Theorem 9, we have $\dim(\bar{\mathfrak{h}}) \leq 3$, so $\dim(\mathfrak{h}) \leq 4$. So we may suppose that $X_8 \notin \mathfrak{h}$. If $\alpha \neq 0$ we have that $\mathfrak{g}_{8,\alpha} \in \mathcal{O}_1$ so Lemma 3(a) implies that $\dim(\mathfrak{h}) \leq 4$. It remains to treat the case $\alpha = 0$. By Remark 3, $\dim(\mathfrak{h}) < 6$. Assume that $\dim(\mathfrak{h}) = 5$. Since $[X_3, X_5] = X_8 \notin \mathfrak{h}$, it follows that $\mathfrak{h}$ cannot have both an element of degree 3 and an element of degree 5. In fact, we can write $\mathfrak{h} = \operatorname{Span}(Y_2, Y_4, Y_5, Y_6, Y_7)$; indeed, we could not have an element Y_3 of degree 3 in $\mathfrak{h}$ since otherwise $\deg([Y_2, Y_3]) = 5$.

It follows that $\mathfrak{h}^\perp$ has an element of degree 2 or 3, by Lemma 1(b). Moreover, $Y_7 = E_7$, by Lemma 1(c). However, if there was $Z \in \mathfrak{h}^\perp$ such that $\deg(Z) = 2$ then $\langle [Z, Y_5], Y_7\rangle \neq 0$ while for $Z \in \mathfrak{h}^\perp$ of degree 3 we would have $\langle [Z, Y_4], Y_7\rangle \neq 0$. In each case we get a contradiction with (1). $\square$

Theorem 11. *Let $\mathfrak{h}$ be a proper totally geodesic subalgebra of $\mathfrak{g}_{9,\alpha}$. Then, for all $\alpha \neq -\frac{5}{2}, -2$, we have* $\dim(\mathfrak{h}) \leq 4$.

Proof. Note that if $\alpha \notin \{-1, \frac{1}{2}\}$, then $\mathfrak{g}_{9,\alpha} \in \mathcal{O}_1$ and Lemma 3(a) implies that $\dim(\mathfrak{h}) \leq 4$ if $X_9 \notin \mathfrak{h}$. Furthermore, $\mathfrak{g}_{9,\alpha}/\operatorname{Span}(X_9) \cong \mathfrak{g}_{8,\alpha}$ and if $\alpha \notin \{-1, 0\}$, then $\mathfrak{g}_{8,\alpha} \in \mathcal{O}_2$ and Lemma 5(b) implies that $\dim(\mathfrak{h}) \leq 4$ if $X_9 \in \mathfrak{h}$. So there are three remaining cases:

(a) $\alpha = 0$ and $X_9 \in \mathfrak{h}$,
(b) $\alpha = -1$,
(c) $\alpha = \frac{1}{2}$ and $X_9 \notin \mathfrak{h}$.

(a) Suppose $\alpha = 0$ and $X_9 \in \mathfrak{h}$. By Lemma 4(b) and Theorem 10, we have $\dim(\mathfrak{h}) \leq 5$. Note that by Lemma 1(b), $\mathfrak{h}$ has no elements of degree 8. First assume that there is a degree 2 element Z in $\mathfrak{h}^\perp$. Then by Lemma 1(d), there are no elements of degree 7 in $\mathfrak{h}$. From $\deg([Y_2, Y_3]) = 5$, $\deg([Y_2, Y_5]) = 7$, $\deg([Y_3, Y_5]) = 8$ for arbitrary elements Y_i of degree $i = 2, 3, 5$, it follows that there exists at most one element $i \in \{2, 3, 5\}$ with Y_i in $\mathfrak{h}$. Hence $\dim(\mathfrak{h}) \leq 4$. Thus we may assume that $E_2 \in \mathfrak{h}$. By Lemma 1(b), there is a degree 3 element Z in $\mathfrak{h}^\perp$, while by Lemma 1(d) there are no elements of degree 6 in $\mathfrak{h}$. Since $\deg([E_2, Y_4]) = 6$, $\deg([Y_3, Y_5]) = 8$, $\deg([E_2, Y_3]) = 5$ for arbitrary elements Y_i of degree $i = 3, 4, 5$, it follows that $\mathfrak{h}$ has no elements of degree 3 and 4. Therefore, $\dim(\mathfrak{h}) \leq 4$.

(b) For $\alpha = -1$, suppose first that $X_9 \in \mathfrak{h}$. Then by Lemma 1(b), $\mathfrak{h}$ has no elements of degree 8. If $Y_2 \in \mathfrak{h}$, then $\dim(\mathfrak{h}) \leq 4$, as all the elements $[Y_2, Y_6]$, $\operatorname{ad}^2(E_2)(Y_4)$ and $\operatorname{ad}^2(Y_3)(E_2)$ have degree 8, so $\mathfrak{h}$ has no elements of degree $3, 4$ and 6. Otherwise, $E_2 \in \mathfrak{h}^\perp$, so by Lemma 1(d), we have no elements of degree 7 in $\mathfrak{h}$. As $\deg([Y_3, Y_5]) = 8$, $\mathfrak{h}$ cannot contain elements of both degree 3 and degree 5, so $\dim(\mathfrak{h}) \leq 4$. Now suppose that $X_9 \notin \mathfrak{h}$. Since $[X_2, X_7] = -X_9$ and $[X_4, X_5] = -X_9$ we have $\dim(\mathfrak{h}) \leq 5$. If $\dim(\mathfrak{h}) = 5$, then $\mathfrak{h}$ has elements Y_3, Y_6, Y_8 of degree 3,6,8 respectively, and in particular, by Lemma 1(c) we may take $Y_8 = E_8 \in \mathfrak{h}$. From Lemma 1(e) we conclude that there are no elements of degree 2 in $\mathfrak{h}^\perp$, and so $E_2 \in \mathfrak{h}$. Hence by Lemma 1(b), there is an element $Z \in \mathfrak{h}^\perp$ of degree 3. However, as $\deg([E_2, Y_3]) = 5$, we obtain a contradiction with Lemma 1(e).

(c) For $\alpha = \frac{1}{2}$ and $X_9 \notin \mathfrak{h}$, the subalgebra $\mathfrak{h}$ has dimension ≤ 5 as $[X_3, X_6] = [X_4, X_5] = \frac{1}{2}X_9$. Suppose $\dim(\mathfrak{h}) = 5$. Arguing as in case (b), $\mathfrak{h}$ is spanned by E_8 and some Y_2, Y_i, Y_j, Y_7, where i is 3 or 6 and j is 4 or 5. First assume that $\mathfrak{h}^\perp$ has an element Z of degree 2. Then by Lemma 1(e), $\mathfrak{h}$ has no elements of degree 6. Thus $i = 3$ and since $\deg([X_2, X_4]) = 6$, we also have $j = 5$. Then $\mathfrak{h}^\perp$ can have no elements of degree $3, 5$ and 6 by Lemma 1(e). It follows that $\mathfrak{h}^\perp$ has an element Z_4 of degree 4. But by Lemma 1(b), $\langle Y_7, E_9\rangle \neq 0$ and so $\langle \nabla_{Y_5} Y_7, Z_4\rangle \neq 0$ contradicting (1). So we may assume there is no elements of degree 2 in $\mathfrak{h}^\perp$. In this case, $E_2 \in \mathfrak{h}$ and by Lemma 1(b), there must be an element $Z_3 \in \mathfrak{h}^\perp$ of degree 3. Then by Lemma 1(e), $\mathfrak{h}$ has no elements of degree 5. Consequently, since $[X_2, X_3] = X_5$, the algebra $\mathfrak{h}$ also has no elements of degree 3. So $\mathfrak{h}$ must have an element of degree 6. However, $\langle Y_7, E_9\rangle \neq 0$ by Lemma 1(b), so $\langle \nabla_{Y_6} Y_7, Z_3\rangle \neq 0$ contradicting (1). This completes the proof of the theorem. $\square$

Theorem 12. *Let* $\mathfrak{h}$ *be a proper totally geodesic subalgebra of* $\mathfrak{g}_{10,\alpha}$*. Then, for all* $\alpha \neq -\frac{5}{2}$*, we have* $\dim(\mathfrak{h}) \leq 5$.

Proof. If $X_{10} \in \mathfrak{h}$ and $\alpha \neq -2$ we may consider the quotient Lie algebra $\mathfrak{g}_{10,\alpha}/\operatorname{Span}(X_{10}) \cong \mathfrak{g}_{9,\alpha}$ and the required result follows from that by Theorem 11.

Suppose that $X_{10} \in \mathfrak{h}$ and $\alpha = -2$. By Lemma 1(b), there are no elements of degree 9 in $\mathfrak{h}$. Note that $[X_3, X_6] = -2X_9$ and $[X_4, X_5] = 3X_9$. If there is an element of degree 2 in $\mathfrak{h}^\perp$, Lemma 1(d) implies that there are no element in $\mathfrak{h}$ of degree 8, from which it follows that $\dim(\mathfrak{h}) \leq 5$. On the other hand, if there are no elements of degree 2 in $\mathfrak{h}^\perp$, there exists an element of degree 3 in $\mathfrak{h}^\perp$, by Lemma 1(b). It follows that there are no elements of degree 7 in $\mathfrak{h}$, again giving that $\dim(\mathfrak{h}) \leq 5$. So we may suppose that $X_{10} \notin \mathfrak{h}$.

If $\alpha \notin \{\frac{1}{2}, -1\}$, then $\mathfrak{g}_{10,\alpha} \in \mathcal{O}_1$, hence by Lemma 3(a) we get $\dim(\mathfrak{h}) \leq 5$. So there are two special cases that remain to be considered: $\alpha = \frac{1}{2}$ and $\alpha = -1$.

By Lemma 4(a), the vector $Y = \pi_{\mathfrak{h}}(X_{10})$ lies in the center of $\mathfrak{h}$. If $Y = 0$, then $X_{10} \in \mathfrak{h}^\perp$, so by Lemma 4(b) and Theorem 11, $\dim(\mathfrak{h}) \leq 4$, as $\mathfrak{g}_{10,\alpha}/\operatorname{Span}(X_{10}) \cong \mathfrak{g}_{9,\alpha}$. Let $m = \deg(Y)$. Suppose that $m = 2$. As Y lies in the center of $\mathfrak{h}$ and as $[X_2, X_3], [X_2, X_4]$ and $[X_2, X_6]$ are all nonzero, $\mathfrak{h}$ contains no elements of degree $3, 4$ or 6, so $\dim(\mathfrak{h}) \leq 5$. Similarly, if $m = 3$, then $\mathfrak{h}$ contains no elements of degree $2, 4$ or 5, and if $m = 4$, then $\mathfrak{h}$ contains no elements of degree $2, 3, 5$ or 6. In both cases, $\dim(\mathfrak{h}) \leq 5$. Furthermore, if $m = 5$ and $\alpha = \frac{1}{2}$, then $\mathfrak{h}$ contains no elements of degree $2, 3$ or 4, so $\dim(\mathfrak{h}) \leq 5$, and if $m = 5$ and $\alpha = -1$, then $\mathfrak{h}$ contains no elements of degree 3 or 4, so $\dim(\mathfrak{h}) \leq 5$ unless $\mathfrak{h} = \operatorname{Span}(Y_2, Y_5, Y_6, Y_7, Y_8, Y_9)$. But then we may take $Y_9 = E_9$ by Lemma 1(b), so $\mathfrak{h}^\perp$ has no elements of degree 2 or 4 by Lemma 1(e), and hence $\mathfrak{h}^\perp$ has at least two linearly independent elements in $\mathfrak{g}_5$, contradicting the fact that $\dim(\mathfrak{h} \cap \mathfrak{g}_5) = 5$ and $\dim(\mathfrak{g}_5) = 6$. Likewise, if $m = 6$ and $\alpha = \frac{1}{2}$, then $\mathfrak{h}$ contains no elements of degree $2, 3$ or 4, so $\dim(\mathfrak{h}) \leq 5$, and if $m = 6$ and $\alpha = -1$, then $\mathfrak{h}$ contains no elements of degree 2 or 4, so $\dim(\mathfrak{h}) \leq 5$ unless $\mathfrak{h} = \operatorname{Span}(Y_3, Y_5, Y_6, Y_7, Y_8, Y_9)$. Then $E_2 \in \mathfrak{h}^\perp$ and we may take $Y_9 = E_9$ by Lemma 1(b). By Lemma 1(e) we get a contradiction with the fact that $Y_7 \in \mathfrak{h}$. Finally, if $m = 7$ and $\alpha = -1$, then $\mathfrak{h}$ contains no elements of degree 2 or 3, so $\dim(\mathfrak{h}) \leq 5$ unless $\mathfrak{h} = \operatorname{Span}(Y_4, Y_5, Y_6, Y_7, Y_8, Y_9)$. But then $E_2, E_3 \in \mathfrak{h}^\perp$ and we may take $Y_9 = E_9$ by Lemma 1(b), which leads to a contradiction with Lemma 1(e), as $Y_7 \in \mathfrak{h}$.

It remains to consider the cases when either $m = 7$ and $\alpha = \frac{1}{2}$ or $m = 8$. Note that in both cases $\mathfrak{h}$ contains the vector $Y = \pi_{\mathfrak{h}}(E_{10})$ of degree m, and then $\mathfrak{h}^\perp$ contains the vector $Z = Y - E_{10}$, also of degree m. Consider two cases.

The Case $\alpha = -1$. Then both $\mathfrak{h}$ and $\mathfrak{h}^\perp$ contain an element of degree 8, say Y_8 and Z_8 respectively. Since $[X_4, X_6] = -[X_3, X_7] = X_{10} \notin \mathfrak{h}$, the subalgebra $\mathfrak{h}$ cannot have elements of both degree 3 and degree 7, and nor can it have elements of both degree 4 and degree 6. So $\dim(\mathfrak{h}) \leq 6$. Suppose that $\dim(\mathfrak{h}) = 6$. Since $[X_2, X_4] = X_6$, we have $\mathfrak{h} = \operatorname{Span}(Y_2, Y_3, Y_5, Y_6, Y_8, E_9)$ or $\mathfrak{h} = \operatorname{Span}(Y_2, Y_5, Y_6, Y_7, Y_8, E_9)$. In both cases, the three-dimensional ideal $(\mathfrak{g}_{10,-1})_8 = \operatorname{Span}(X_8, X_9, X_{10}) \subset \mathfrak{g}_{10,-1}$ contains linearly independent vectors $Y_8, E_9 \in \mathfrak{h}$ and $Z_8 \in \mathfrak{h}^\perp$, hence $(\mathfrak{g}_{10,-1})_8 = \operatorname{Span}(Y_8, E_9, Z_8)$. Then by Lemma 4(b), there is an inner product on the algebra $\mathfrak{g}_{10,-1}/(\mathfrak{g}_{10,-1})_8 \cong \mathfrak{g}_{7,-1}$, for which $\overline{\mathfrak{h}} = \pi(\mathfrak{h})$ is a totally geodesic subalgebra. But $\dim(\overline{\mathfrak{h}}) = 4$, which contradicts Theorem 9.

The Case $\alpha = \frac{1}{2}$. Then both $\mathfrak{h}$ and $\mathfrak{h}^\perp$ contain either an element of degree 7 or an element of degree 8.

As $[X_4, X_6] = \frac{1}{2}X_{10} \notin \mathfrak{h}$ and $E_1 \in \mathfrak{h}^\perp$, we have $\dim(\mathfrak{h}) \leq 7$. Suppose first that $\dim(\mathfrak{h}) = 7$. Since $\mathfrak{h}$ does not have both an element of degree 4 and an element of degree 6, and since $[X_2, X_4] = \frac{5}{2}X_6$, we may write $\mathfrak{h} = \operatorname{Span}(Y_2, Y_3, Y_5, \dots, Y_9)$. But then by Lemma 2, $\mathfrak{h}^\perp$ has no elements of degree ≥ 7, a contradiction.

Now suppose that $\dim(\mathfrak{h}) = 6$. We consider three subcases:

(i) there are no elements of degree 3 in $\mathfrak{h}$,
(ii) there is an element Y_3 of degree 3 but no elements of degree 2 in $\mathfrak{h}$,
(iii) there are elements $Y_2, Y_3 \in \mathfrak{h}$ of degree 2 and 3 respectively.

Subcase (i): Arguing as above, since $\dim(\mathfrak{h}) = 6$, $[X_2, X_4] = \frac{5}{2}X_6$ and $[X_4, X_6] = \frac{1}{2}X_{10} \notin \mathfrak{h}$, we may write $\mathfrak{h} = \operatorname{Span}(Y_2, Y_5, Y_6, Y_7, Y_8, E_9)$, so by Lemma 2, $\mathfrak{h}^\perp$ has no elements of degree 7 or 8, a contradiction.

Subcase (ii): We have $E_2 \in \mathfrak{h}^\perp$. As $\dim(\mathfrak{h}) = 6$ we have $\mathfrak{h} = \operatorname{Span}(Y_3, Y_4, Y_5, Y_7, Y_8, E_9)$ or $\mathfrak{h} = \operatorname{Span}(Y_3, Y_5, Y_6, Y_7, Y_8, E_9)$. In the latter case, $\mathfrak{h}^\perp$ has no elements of degree 7 or 8, by Lemma 2, a contradiction. In the former case, suppose for the moment that there exists $Z_8 \in \mathfrak{h}^\perp$. As $\operatorname{Span}(Y_8, E_9, Z_8) = \operatorname{Span}(E_8, E_9, E_{10})$, we can choose $Y_7 = E_7$ and then $Y_5 \in \operatorname{Span}(E_5, E_6)$, so $E_5 \in \mathfrak{h}$, by Lemma 1(c). But since $[X_2, X_7] = [X_2, X_8] = [X_3, X_7] = 0$, we have $[E_2, E_7] = 0$ and hence $2\langle \nabla_{E_5} E_7, E_2\rangle = \langle [E_2, E_5], E_7\rangle \neq 0$, contradicting (1). It follows that $\mathfrak{h}^\perp$ has no elements of degree 8, and therefore, by Lemma 1(e), we have $\mathfrak{h}^\perp = \operatorname{Span}(E_1, E_2, Z_3, Z_7)$. Arguing as before, we obtain $E_5 \in \mathfrak{h}$. We have $Y_4 := E_4 + a_{4,6}E_6 + a_{4,10}E_{10}$ for some $a_{4,6}, a_{4,10} \in \mathbb{R}$. Then $\langle Y_4, Z_7\rangle = 0$ implies $a_{4,10} = 0$. Then by Lemma 1(b), $a_{4,6} \neq 0$ as $E_5 \in \mathfrak{h}$. However, $\langle \nabla_{Y_4} Y_4, E_2\rangle = a_{4,6}\langle [E_2, E_4], E_6\rangle \neq 0$, contradicting (1).

Subcase (iii): Here $\mathfrak{h}$ contains an element $[Y_2, Y_3]$ of degree 5 and elements $\operatorname{ad}^2(Y_2)(Y_3)$ and $\operatorname{ad}^2(Y_3)(Y_2)$ of degree 7 and 8 respectively. Moreover, $\mathfrak{h}$ has no elements of degree 4, by the same argument used in subcase (i), and no element Y_6 of degree 6, as otherwise $\mathfrak{h}$ would also contain an element $[Y_3, Y_6]$ of degree 9 contradicting the fact that $\dim(\mathfrak{h}) = 6$. So $\mathfrak{h} = \operatorname{Span}(Y_2, Y_3, Y_5, Y_7, Y_8, E_9)$. We may assume that $Y_5 := E_5 + a_{5,6}E_6 + a_{5,10}E_{10}$ and $Y_i := E_i + a_{i,10}E_{10}$, $i = 7, 8$. By Lemma 1(e) there are no elements of degree 4 or 6 in $\mathfrak{h}^\perp$. Moreover, by the same argument used in subcase (ii), $\mathfrak{h}^\perp$ has no elements of degree 8. It follows that $\mathfrak{h}^\perp$ contains an element Z_7 of degree 7. Then we have $E_5 \in \mathfrak{h}$ and so $\mathfrak{h}^\perp = \operatorname{Span}(E_1, Z_2, Z_3, Z_7)$. Moreover, without loss of generality we may assume that $\langle Y_3, E_j\rangle = 0$ for $j = 7, 8, 9, 10$. The contradiction with (1) is then obtained from

$$0 = 2\langle \nabla_{Y_3} E_5, Z_2\rangle = \langle [Z_2, Y_3], E_5\rangle + \langle [Z_2, E_5], Y_3\rangle = \langle [Z_2, Y_3], E_5\rangle.$$

□

Theorem 13. *Let $\mathfrak{h}$ be a proper totally geodesic subalgebra of $\mathfrak{g}_{11,\alpha}$. Then, for all $\alpha \notin \{-\frac{5}{2}, -1, -3\}$, we have* $\dim(\mathfrak{h}) \leq 5$.

Proof. Consider two cases: $X_{11} \in \mathfrak{h}$ and $X_{11} \notin \mathfrak{h}$.

Case $X_{11} \in \mathfrak{h}$: Consider the quotient map $\pi : \mathfrak{g}_{11,\alpha} \to \mathfrak{g}_{11,\alpha}/\operatorname{Span}(X_{11})$. For $\alpha \notin \{-2, 0, \frac{1}{2}\}$, we have $\mathfrak{g}_{11,\alpha}/\operatorname{Span}(X_{11}) \cong \mathfrak{g}_{10,\alpha} \in \mathcal{O}_2$, and Lemma 5(b) gives $\dim(\mathfrak{h}) \leq 5$. Let $\alpha \in \{-2, 0, \frac{1}{2}\}$. Lemma 4(b) and Theorem 12 give $\dim(\pi(\mathfrak{h})) \leq 5$ and so $\dim(\mathfrak{h}) \leq 6$. Assume $\dim(\mathfrak{h}) = 6$. Note that $\mathfrak{h}$ has no elements of degree 10 by Lemma 1(b).

If $\alpha \neq \frac{1}{2}$, then $[X_2, X_8], [X_3, X_7], [X_4, X_6]$ are all nonzero multiplies of X_{10}, so $\mathfrak{h}$ cannot have elements of both degree 2 and degree 8, nor can it have elements of both degree 3 and degree 7, nor of both degree 4 and degree 6. Then $\mathfrak{h}$ necessarily contains elements Y_5, Y_9. By Lemma 1(c) we may take $Y_9 = E_9$. Then by Lemma 1(d), $\mathfrak{h}^\perp$ has no elements of degree

2, and so we have $E_2 \in \mathfrak{h}$. Consequently, $\mathfrak{h}$ has no elements of degree 8. Since $Y_5 \in \mathfrak{h}$ and $[X_3, X_5] = X_8,\ [X_2, X_5] = (1+\alpha)X_7$, we conclude that $\mathfrak{h}$ has an element of degree 7, but no elements of degree 3. It follows that $\mathfrak{h} = \mathrm{Span}(E_2, Y_j, Y_5, Y_7, E_9, E_{11})$, where $j = 4$ or 6.

First suppose $\alpha = -2$. Since $[X_2, X_6] = -2X_8$ and $\mathfrak{h}$ has no elements of degree 8, $\mathfrak{h}$ has no elements of degree 6. Hence $\mathfrak{h}$ has an element of degree 4. Then by Lemma 1(d), $\mathfrak{h}^\perp$ has no elements of degree 2, 4, 6, 7 or 9. So we can write $\mathfrak{h}^\perp = \mathrm{Span}(E_1, E_3, Z_5, Z_8, E_{10})$. But then since $[X_4, X_5] = 3X_9$, we have $\langle \nabla_{Y_4} E_9, Z_5\rangle \neq 0$, contradicting (1).

Now suppose $\alpha = 0$. Since $\mathfrak{h}$ does not have elements of both degree 4 and 6, and since $[X_2, X_4] = 2X_6$, the subalgebra $\mathfrak{h}$ has no elements of degree 4, and hence it must have one of degree 6. So we have $\mathfrak{h} = \mathrm{Span}(E_2, Y_5, Y_6, Y_7, E_9, E_{11})$. But since $E_3 \in \mathfrak{h}^\perp$ and $[X_3, X_6]$ is a nonzero multiple of X_9, we have $\langle \nabla_{Y_6} E_9, E_3\rangle \neq 0$, again contradicting (1).

Now suppose $\alpha = \frac{1}{2}$. Note that $\mathfrak{h}$ does not have elements of both degree 4 and 6. First assume that $\mathfrak{h}^\perp$ has no elements of degree 2. So $E_2 \in \mathfrak{h}$. Note that $\mathfrak{h}^\perp$ has an element Z_3, as otherwise we would have $E_2, E_3 \in \mathfrak{h}$, contradicting Lemma 1(b). But then as $[X_3, X_8]$ is a nonzero multiple of X_{11}, $\mathfrak{h}$ has no elements of degree 8 by Lemma 1(d). Since $[X_2, X_4] = \frac{5}{2}X_6,\ [X_2, X_6] = \frac{1}{2}X_8$ and $[X_2, X_3] = \frac{5}{2}X_5,\ [X_3, X_5] = X_8$, we conclude that $\mathfrak{h}$ has no elements of degree $3, 4, 6$ or 8, giving $\dim(\mathfrak{h}) \leq 5$. So we may suppose that $\mathfrak{h}^\perp$ has an element Z_2. It then follows from Lemma 1(d) that $\mathfrak{h}$ has no elements of degree 9. Consequently $\mathfrak{h}$ does not have elements of both degree 3 and 6, and nor does it have elements of both degree 4 and 5. Then $\mathfrak{h}$ necessarily has elements Y_2, Y_7, Y_8 and has no elements of degree 4, as $\mathrm{ad}^2(X_4)(X_2) = -\frac{5}{4}X_{10}$. We conclude that either $\mathfrak{h} = \mathrm{Span}(Y_2, Y_3, Y_5, Y_7, Y_8, E_{11})$ or $\mathfrak{h} = \mathrm{Span}(Y_2, Y_5, Y_6, Y_7, Y_8, E_{11})$. Then by Lemma 1(d), we have $\mathfrak{h}^\perp = \mathrm{Span}(E_1, Z_2, Z_5, Z_7, E_{10})$ or $\mathfrak{h}^\perp = \mathrm{Span}(E_1, Z_2, Z_7, Z_8, E_{10})$ respectively. For an ideal $(\mathfrak{g}_{11,\frac{1}{2}})_{10} = \mathrm{Span}(E_{10}, E_{11})$, we have $\mathfrak{g}_{11,\frac{1}{2}}/(\mathfrak{g}_{11,\frac{1}{2}})_{10} \cong \mathfrak{g}_{9,\frac{1}{2}}$. But then by Lemma 4(b), the image of $\mathfrak{h}$ in $\mathfrak{g}_{9,\frac{1}{2}}$ is a totally geodesic subalgebra of dimension 5, contradicting Theorem 11. This completes the proof in the case $X_{11} \in \mathfrak{h}$.

Case $X_{11} \notin \mathfrak{h}$: Let $\alpha_1 \approx -1.5919$ and $\alpha_2 \approx 1.5342$ be the (unique) real roots of the polynomials $2\alpha^3 + 2\alpha^2 + 3$ and $4\alpha^3 + 8\alpha^2 - 8\alpha - 21$ respectively. By Lemma 3(a), the required result holds provided $\alpha \neq -\frac{1}{4}, \alpha_1, \alpha_2$. In these special cases, $\dim(\mathfrak{h}) \leq 6$, as $\mathfrak{h}$ cannot contain elements of both degree i and $11 - i$ when $[X_i, X_{11-i}]$ is a nonzero multiple of X_{11}. So, suppose that $\dim(\mathfrak{h}) = 6$. Then $\mathfrak{h}$ must necessarily have an element Y_{10} and we may take $Y_{10} = E_{10}$ by Lemma 1(c).

First suppose $\alpha = -\frac{1}{4}$. Then $\mathfrak{h} = \mathrm{Span}(Y_5, Y_6, E_{10}, Y_i, Y_j, Y_k)$, where $i \in \{2, 9\}$, $j \in \{3, 8\}$, $k \in \{4, 7\}$. If there is an element $Z_2 \in \mathfrak{h}^\perp$, then by Lemma 1(e), there are no elements of degree 8 in $\mathfrak{h}$, and so we have an element $Y_3 \in \mathfrak{h}$. But then $\mathfrak{h}$ contains an element $[Y_3, Y_5]$ of degree 8, a contradiction. On the other hand, if $E_2 \in \mathfrak{h}$, then $\mathfrak{h}$ contains an element $[E_2, Y_5]$ of degree 7. But then $\mathfrak{h}^\perp$ has no elements of degree 3 by Lemma 1(e), so $E_2, E_3 \in \mathfrak{h}$, which contradicts Lemma 1(b).

Now suppose $\alpha = \alpha_1$, so that $2\alpha^3 + 2\alpha^2 + 3 = 0$. Then there are elements $Y_2, Y_9 \in \mathfrak{h}$. If there is an element $Z_2 \in \mathfrak{h}^\perp$, then there are no elements of degree 8 in $\mathfrak{h}$ by Lemma 1(e), so there is necessarily an $Y_3 \in \mathfrak{h}$. But then $\mathfrak{h}$ contains an element $\mathrm{ad}^2(Y_3)(E_2)$ of degree 8, a contradiction. On the other hand, if $\mathfrak{h}^\perp$ has no elements of degree 2, then $E_2 \in \mathfrak{h}$, so by Lemma 1(b), there is an element $Z_3 \in \mathfrak{h}^\perp$. Then $\mathfrak{h}$ has no elements of degree 7 by Lemma 1(e), hence no elements of degree 3 or 5, as $\mathrm{ad}^2(X_2)(X_3)$ and $\mathrm{ad}(X_2)(X_5)$ are nonzero multiples

of X_7. Then $\mathfrak{h} = \operatorname{Span}(E_2, Y_4, Y_6, Y_8, Y_9, E_{10})$. We may take $Y_9 = E_9 + aE_{11}$. Then $a \neq 0$ by Lemma 1(b), and in particular, $X_{11} \notin \mathfrak{h}^\perp$. It follows that $\mathfrak{h}^\perp = \operatorname{Span}(E_1, Z_3, Z_5, Z_7, Z_9)$. But as $[X_5, X_6]$ is a nonzero multiple of X_{11}, we have $2\langle \nabla_{Y_6}(E_9 + aE_{11}), Z_5\rangle = a\langle [Z_5, Y_6], E_{11}\rangle \neq 0$, contradicting (1).

Finally, suppose $\alpha = \alpha_2$, so that $4\alpha^3 + 8\alpha^2 - 8\alpha - 21 = 0$. Then there exist $Y_3, Y_8 \in \mathfrak{h}$. By Lemma 1(e), $\mathfrak{h}^\perp$ has no elements of degree 2, so $E_2 \in \mathfrak{h}$. But then $\mathfrak{h}$ contains an element $\operatorname{ad}^4(E_2)(Y_3)$ of degree 11, which is a contradiction. $\square$

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Department of Mathematics and Statistics, La Trobe University, Melbourne, Australia 3086

E-mail address: `G.Cairns@latrobe.edu.au`

E-mail address: `ahinicgalic@students.latrobe.edu.au`

E-mail address: `Y.Nikolayevsky@latrobe.edu.au`